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Analysis of Pumping Test Data From Anisotropic Unconfined Aquifers Considering Delayed Gravity Response

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Previously a new analytical model was proposed by the author for the delayed response process characterizing flow to a well in an unconfined aquifer. The new approach differs from that of Boulton (1954, 1963, 1970, 1973) and Boulton and Pontin (1971) in that it is based only on well-defined physical parameters of the aquifer system. As such, it can be used to develop methods for determining the hydraulic properties of anisotropic unconfined aquifers from field drawdown data. Two methods of analysis are described, one based on the matching of field data with theoretical type curves and the other based on the semilogarithmic relationship between drawdown and time. Owing to the reversible nature of the delayed response process as represented by the analytical model, data from recovery tests can be used to determine aquifer transmissivity. All of these methods are illustrated by applying them to pumping test data collected by the French Bureau de Recherches Géologiques et Minières in Gironde (Bonnet et al., 1970). Similar procedures can be used to analyze data from partially penetrating wells, but this method requires that a special set of theoretical curves be developed for each field situation. Such theoretical curves can easily be developed with the aid of a computer program available from the author. An explicit mathematical relationship is derived between Boulton's (1963) delay index, $1/\alpha$, and the physical characteristics of the aquifer. It is shown that contrary to the assumption of Boulton the parameter α is not a characteristic constant of the aquifer but decreases linearly with the logarithm of r, the radial distance from the pumping well. This discovery makes it possible to reinterpret the results of pumping tests that were previously obtained with the aid of Boulton's theory without necessarily reexamining the original drawdown data. Results from pumping tests performed by the Bureau de Recherches Géologiques et Minières in Gironde and by Prickett (1965) in Illinois are used to illustrate this last point.

In analyzing pumping test data from unconfined aquifers one often finds that the drawdowns vary at different rates from those predicted by the traditional *Theis* [1935] equation. When these drawdowns are plotted versus time on logarithmic paper, they usually delineate an S-shaped curve consisting of a steep segment at early times, a flat segment at intermediate times, and a somewhat steeper segment at later times. The physical phenomenon that causes this behavior is known as delayed yield, delayed drainage, or delayed gravity response.

A semiempirical mathematical model capable of reproducing all three segments of the time-drawdown curve in an unconfined aquifer was introduced into the literature as early as 1954 by *Boulton* [1954, 1963]. In his model, Boulton assumed that the amount of water released from storage per unit horizontal area of the aquifer due to a unit drawdown occurring at time τ is the sum of two components: S, a volume of water instantaneously released at time τ, and S_y, a volume of water the release of which is delayed in time according to the empirical formula $\alpha S_y \exp[-\alpha(t - \tau)]$, where t $(t \geq \tau)$ is time and α is a characteristic constant of the aquifer. This model was later extended by *Boulton* [1970] and *Boulton and Pontin* [1971] to account for anisotropy and the effect of vertical flow components in the aquifer.

Boulton [1963] used his theory to develop two complementary sets of type curves, known as type A and type B curves, corresponding to the particular case where $\sigma = S/S_y$ approaches zero. Boulton recommended that type A curves be used in the interpretation of early drawdown data and that type B curves be used in the interpretation of late drawdown data. *Prickett* [1965] described in detail a graphical procedure, based on Boulton's type curves, for determining the transmissivity T, the storage coefficients S and S_y, and the so-called delayed index $1/\alpha$ of an unconfined aquifer and applied it to 18 pumping tests throughout the United States. *Bonnet et al.* [1970] described how the method had been applied to an unconfined situation in Gironde, France.

Another approach was taken by *Berkaloff* [1963], who had found by plotting the drawdown versus the time on semilogarithmic paper that part of the early and the late data tend to fall on two parallel straight lines, whereas part of the intermediate data tend to fall on a horizontal line. Berkaloff's method enables one to calculate the values of T, S, S_y, and α in an unconfined aquifer based on the theoretical results of Boulton. An account of this method was given by *Bonnet et al.* [1970], who also showed how it had been applied to two pumping tests, one in Gironde, France, and the other in Boulanouar, Mauritania.

In a later work, *Boulton* [1970] advocated the use of distance-drawdown analyses for the determination of transmissivities and storage coefficients as a supplement for the aforementioned time-drawdown method of determining the delay index. *Prickett* [1965, p. 9], on the other hand, claimed that such analyses should be applied only after the effect of delayed yield has dissipated in all the observation wells.

A method for determining the parameters of an incompressible anisotropic unconfined aquifer from data obtained under conditions of partial penetration was developed by *Dagan* [1967*a*, *b*] and was applied in the field by *Degallier* [1969] and *Ramon* [1970]. Dagan's method does not take into account the phenomenon of delayed gravity response, and therefore it is limited in its application to relatively large distances from the pumping well and to sufficiently large values of time, at which the effect of elastic storage is very small [*Neuman*, 1972, 1973, 1974]. The method is based on the matching of measured data with theoretical type curves, the shape of which varies from one field situation to another. Thus

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a different set of type curves must be developed for each particular observation well within any given pumping test setup.

In the past, the phenomenon of delayed yield has often been attributed to unsaturated flow above the water table. Recently, *Neuman* [1972, 1973, 1974] showed that this phenomenon can be simulated mathematically by using constant values of specific storage and specific yield without recourse to unsaturated flow theory. The new model treats the unconfined aquifer as a compressible system and the phreatic surface as a moving material boundary. It differs from that of *Boulton* [1954, 1963, 1970, 1973] and *Boulton and Pontin* [1971] in that it is based only on well-defined physical parameters of the aquifer and no longer involves such semiempirical quantities as Boulton's delay index $1/\alpha$. The new model takes into account aquifer anisotropy and enables one to investigate the effect of partially penetrating wells on the drawdown when delayed gravity response (or delayed yield) is important [*Neuman*, 1974].

The purpose of the present paper is to show how the new theory of *Neuman* [1972, 1973, 1974] can be used to determine the hydraulic characteristics of anisotropic unconfined aquifers from pumping test data. A distinction is made between the case in which the pumping well and the observation well are perforated throughout the saturated thickness of the aquifer and the case in which at least one of these wells is partially penetrating. Pumping test data which were previously analyzed by *Prickett* [1965] and *Bonnet et al.* [1970] using Boulton's theory are reinterpreted in light of the new methods proposed herein. A mathematical relationship between Boulton's delay index $1/\alpha$ and other measurable physical parameters of the aquifer is derived. This relationship shows that contrary to the assumption of Boulton, α is not a characteristic constant of the aquifer but decreases linearly with the logarithm of r, the radial distance from the pumping well.

COMPLETE PENETRATION

Type curve method. When the pumping well and the observation well are perforated throughout the entire saturated thickness of the aquifer, the drawdown in the observation well is given by [*Neuman*, 1973, equations 1–6]

$$s(r, t) = \frac{Q}{4\pi T}\int_0^\infty 4y\, J_0(y\beta^{1/2})[u_0(y) + \sum_{n=1}^{\infty} u_n(y)]\, dy \quad (1)$$

where

$$u_0(y) = \frac{\{1 - \exp[-t_s\beta(y^2 - \gamma_0^2)]\}\tanh(\gamma_0)}{\{y^2 + (1+\sigma)\gamma_0^2 - [(y^2 - \gamma_0^2)^2/\sigma]\}\gamma_0} \quad (2)$$

$$u_n(y) = \frac{\{1 - \exp[-t_s\beta(y^2 + \gamma_n^2)]\}\tan(\gamma_n)}{\{y^2 - (1+\sigma)\gamma_n^2 - (y^2 + \gamma_n^2)^2/\sigma\}\gamma_n} \quad (3)$$

and the terms γ_0 and γ_n are the roots of the equations

$$\sigma\gamma_0 \sinh(\gamma_0) - (y^2 - \gamma_0^2)\cosh(\gamma_0) = 0 \quad (4)$$

$$\gamma_0^2 < y^2$$

$$\sigma\gamma_n \sin(\gamma_n) + (y^2 + \gamma_n^2)\cos(\gamma_n) = 0 \quad (5)$$

$$(2n - 1)(\pi/2) < \gamma_n < n\pi \qquad n \geq 1$$

Equation (1) is expressed in terms of three independent dimensionless parameters σ, β, and t_s or t_y, where the dimensionless time parameters are related to each other by $t_y = \sigma t_s$. This relatively large number of dimensionless parameters makes it practically impossible to construct a sufficient number of type curves to cover the entire range of values necessary for field application. For a set of type curves to be useful, they are normally expressed in terms of not more than two independent dimensionless parameters.

A convenient way to reduce the number of independent dimensionless parameters from three to two is to consider the case in which σ approaches zero, i.e., in which S is much less than S_y. The results are two asymptotic families of type curves that in analogy to *Boulton* [1963] and *Prickett* [1965] will be referred to as type A and type B curves. The curves are shown in Figure 1, and the corresponding numerical values are given in Table 1. The numerical approach used in obtaining these values from (1) has been described in an earlier work [*Neuman*, 1972, Appendix D]. A listing of the computer program complete with users' instructions is available from the author on request.

The curves lying to the left of the values of β in Figure 1 are called type A curves and correspond to the top scale expressed in terms of t_s. The curves lying to the right of the values of β in Figure 1 are called type B curves and correspond to the bottom scale expressed in terms of t_y. The Theis curves with respect to both dimensionless time parameters t_s and t_y have been included in the figure for reference purposes. Type A curves are intended for use with early drawdown data and type B curves with late drawdown data.

Both families of type curves approach a set of horizontal asymptotes the lengths of which depend on the value of σ. When σ tends to zero, the two families of type curves are removed from each other an infinite distance [*Neuman*, 1972, Figures 2 and 3], and for this reason they must be plotted on different scales, one with respect to t_s and the other with respect to t_y.

In describing how the type curves are used to determine aquifer parameters, we shall follow a methodology similar to that previously outlined by *Prickett* [1965, pp. 7–9] in connection with Boulton's theory. According to this approach, one first plots the drawdown s at a given observation well on logarithmic paper against the values of time t. The rest of the procedure consists of the following three steps:

1. One first superimposes the field data on the type B curves, keeping the vertical and the horizontal axes of both graphs parallel to each other and matching as much of the latest time-drawdown data to a particular type curve as one can. The value of β corresponding to this type curve is noted, and a match point is chosen anywhere on the overlapping portion of the two sheets of paper. The coordinates of this match point are s^* and s_D^* along the vertical axis and t^* and t_y^* along the horizontal axis. The transmissivity is now calculated from

$$T = c_1(Qs_D^*/s^*) \quad (6)$$

and the specific yield from

$$S_y = c_2(Tt^*/r^2t_y^*) \quad (7)$$

where c_1 and c_2 are constants depending on the units. If a consistent set of units such as cgs is used, $c_1 = 1/4\pi = 0.0796$, and $c_2 = 1.0$. If Q is expressed in gallons per minute, and the gallon-day-foot system is used for all other quantities, then $c_1 = 114.6$, and $c_2 = 0.1337$.

2. Next one superimposes the field data on the type A curves, keeping the vertical and the horizontal axes of both graphs parallel to each other and matching as much of the earliest time-drawdown data to a particular type curve as one can. The value of β corresponding to this type curve must be

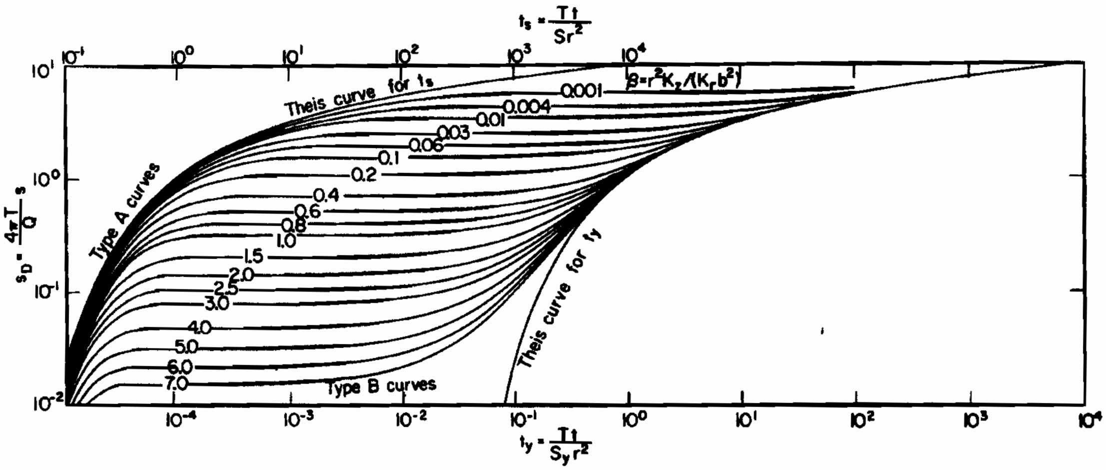

Fig. 1. Type curves for fully penetrating wells.

the same as that obtained earlier from the type B curves. A new match point is selected on the overlapping portion of the two sheets of paper, and its coordinates s^*, s_D^*, t^*, and t_s^* are noted. The transmissivity is again calculated from (6), and its value should be approximately equal to that previously calculated from the late drawdown data. The elastic storage coefficient is obtained from

$$S = c_2(Tt^*/r^2t_s^*) \tag{8}$$

where c_2 is the same as it is in (7).

3. Having determined the transmissivity of the aquifer, one can calculate the horizontal permeability

$$K_r = T/b \tag{9}$$

The degree of anisotropy K_D is obtained from the value of β according to

$$K_D = \beta b^2/r^2 \tag{10}$$

By knowing the values of K_r and K_D, one can determine the vertical permeability using the relationship

$$K_z = K_D K_h \tag{11}$$

The parameter σ is calculated from

$$\sigma = S/S_y \tag{12a}$$

and the specific storage of the aquifer from

$$S_s = S/b \tag{12b}$$

After having determined all the aquifer parameters, one now has the option of obtaining the complete type curve for his field data by integrating (1) with the aid of the computer program mentioned earlier. A graphical procedure for accomplishing the same result has been described by *Prickett* [1965, p. 8]. According to this approach, the complete type curve is obtained by joining the appropriate type A curve and type B curve by a straight line that is tangent to both curves. Many of the intermediate field data that might not have been included in matching the early or late data should fall along the straight line. The slope of this line will be essentially zero for all values of σ below 10^{-2}. Prickett found that type curves obtained by his graphical procedure differ only little from Boulton's theoretical curves for values of σ as large as about 0.2, which covers essentially the entire range of practical interest. The shape of our curves is very similar to that of Boulton's, and therefore Prickett's method is applicable to both sets of curves.

Another way to match the field data with the theoretical curves is to plot the type A and type B curves on two separate sheets of transparent logarithmic paper. This makes it possible to superimpose both type A and type B curves on the field data at the same time.

Semilogarithmic method. When the data from Table 1 are replotted on semilogarithmic paper, the result is that shown in Figure 2. It is seen that the late drawdown data tend to fall on a straight line that according to *Jacob* [1950] is given by

$$s_D = 2.303 \log_{10} (2.246t_y) \tag{13a}$$

The intermediate data tend to fall on a horizontal line, whereas some of the early data tend to fall near the line

$$s_D = 2.303 \log_{10} (2.246t_s) \tag{13b}$$

Let $t_{y\beta}$ be the value of t_y corresponding to the intersection of any horizontal line with the inclined line described by (13*a*). For example, Figure 2 shows that the value of $t_{y\beta}$ for $\beta = 0.03$ is equal to 5.2. When the values of $1/\beta$ are plotted versus $t_{y\beta}$ on logarithmic paper, the result is a unique curve shown in Figure 3. The values from which this curve was plotted were obtained with the aid of Table 1 and (13*a*) and are given in Table 2. A good approximation for the relationship between β and $t_{y\beta}$ within the range $4.0 \leq t_{y\beta} \leq 100.0$ is given by the equation

$$\beta = \frac{0.195}{t_{y\beta}^{1.1053}} \qquad 4.0 \leq t_{y\beta} \leq 100.0 \tag{14}$$

which is represented by the dashed line in Figure 3.

In describing how this is used to determine aquifer parameters we shall follow a methodology somewhat similar to that previously outlined by *Berkaloff* [1963] in connection with Boulton's theory. According to this approach, one first plots the drawdown s at a given observation well on semilogarithmic paper against the values of time t. The rest of the procedure consists of the following four steps:

1. A straight line is fitted to the late portion of the time-drawdown data. The intersection of this line with the horizontal axis corresponding to $s = 0$ is denoted by t_L. The change in s along this line corresponding to a tenfold increase in t (i.e., to

TABLE 1*a*. Values of s_D for the Construction

t_s	$\beta = 0.001$	$\beta = 0.004$	$\beta = 0.01$	$\beta = 0.03$	$\beta = 0.06$	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0.6$
1×10^{-1}	2.48×10^{-2}	2.43×10^{-2}	2.41×10^{-2}	2.35×10^{-2}	2.30×10^{-2}	2.24×10^{-2}	2.14×10^{-2}	1.99×10^{-2}	1.88×10^{-2}
2×10^{-1}	1.45×10^{-1}	1.42×10^{-1}	1.40×10^{-1}	1.36×10^{-1}	1.31×10^{-1}	1.27×10^{-1}	1.19×10^{-1}	1.08×10^{-1}	9.88×10^{-2}
3.5×10^{-1}	3.58×10^{-1}	3.52×10^{-1}	3.45×10^{-1}	3.31×10^{-1}	3.18×10^{-1}	3.04×10^{-1}	2.79×10^{-1}	2.44×10^{-1}	2.17×10^{-1}
6×10^{-1}	6.62×10^{-1}	6.48×10^{-1}	6.33×10^{-1}	6.01×10^{-1}	5.70×10^{-1}	5.40×10^{-1}	4.83×10^{-1}	4.03×10^{-1}	3.43×10^{-1}
1×10^{0}	1.02×10^{0}	9.92×10^{-1}	9.63×10^{-1}	9.05×10^{-1}	8.49×10^{-1}	7.92×10^{-1}	6.88×10^{-1}	5.42×10^{-1}	4.38×10^{-1}
2×10^{0}	1.57×10^{0}	1.52×10^{0}	1.46×10^{0}	1.35×10^{0}	1.23×10^{0}	1.12×10^{0}	9.18×10^{-1}	6.59×10^{-1}	4.97×10^{-1}
3.5×10^{0}	2.05×10^{0}	1.97×10^{0}	1.88×10^{0}	1.70×10^{0}	1.51×10^{0}	1.34×10^{0}	1.03×10^{0}	6.90×10^{-1}	5.07×10^{-1}
6×10^{0}	2.52×10^{0}	2.41×10^{0}	2.27×10^{0}	1.99×10^{0}	1.73×10^{0}	1.47×10^{0}	1.07×10^{0}	6.96×10^{-1}	
1×10^{1}	2.97×10^{0}	2.80×10^{0}	2.61×10^{0}	2.22×10^{0}	1.85×10^{0}	1.53×10^{0}	1.08×10^{0}		
2×10^{1}	3.56×10^{0}	3.30×10^{0}	3.00×10^{0}	2.41×10^{0}	1.92×10^{0}	1.55×10^{0}			
3.5×10^{1}	4.01×10^{0}	3.65×10^{0}	3.23×10^{0}	2.48×10^{0}	1.93×10^{0}				
6×10^{1}	4.42×10^{0}	3.93×10^{0}	3.37×10^{0}	2.49×10^{0}	1.94×10^{0}				
1×10^{2}	4.77×10^{0}	4.12×10^{0}	3.43×10^{0}	2.50×10^{0}					
2×10^{2}	5.16×10^{0}	4.26×10^{0}	3.45×10^{0}						
3.5×10^{2}	5.40×10^{0}	4.29×10^{0}	3.46×10^{0}						
6×10^{2}	5.54×10^{0}	4.30×10^{0}							
1×10^{3}	5.59×10^{0}								
2×10^{3}	5.62×10^{0}								
3.5×10^{3}	5.62×10^{0}	4.30×10^{0}	3.46×10^{0}	2.50×10^{0}	1.94×10^{0}	1.55×10^{0}	1.08×10^{0}	6.96×10^{-1}	5.07×10^{-1}

Values were obtained from (1) by setting $\sigma = 10^{-9}$.

one logarithmic cycle) is denoted by Δs_L. Then according to (13*a*) the transmissivity of the aquifer is determined from

$$T = c_3(Q/\Delta s_L) \tag{15}$$

and the specific yield from

$$S_y = c_4(Tt_L/r^2) \tag{16}$$

where c_3 and c_4 are constants depending on the units. If a consistent set of units is used, $c_3 = 2.303/4\pi = 0.1833$, and $c_4 = 2.246$. If Q is expressed in gallons per minute and the gallon-day-foot system is used for all other quantities, then $c_3 = 263.9$, and $c_4 = 0.3003$.

2. A horizontal line is fitted to the intermediate portion of the time-drawdown data. The value of t corresponding to the intersection of this horizontal line with the straight line passing through the late data is denoted by t_β. Knowing T and S_y from step 1, one computes the dimensionless time $t_{y\beta}$ with the formula

$$t_{y\beta} = c_2(Tt_\beta/S_yr^2) \tag{17}$$

where c_2 is the same as it is in (7). The value of β can now be obtained directly from the curve in Figure 3 or, for a limited range of $t_{y\beta}$ values, from (14).

3. A straight line is fitted to a portion of the early time-drawdown data. If the slope of this line differs markedly from that of the line passing through the late data, step 3 must be skipped, and in this case S must be determined by the type curve method. If the two lines are nearly parallel to each other, the intersection of the early line with the horizontal axis at $s = 0$ is denoted by t_E. The change in s along this line, corresponding to a tenfold increase in t, is denoted by Δs_E. The transmissivity is then calculated from

TABLE 1*b*. Values of s_D for the Construction

t_y	$\beta = 0.001$	$\beta = 0.004$	$\beta = 0.01$	$\beta = 0.03$	$\beta = 0.06$	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0.6$
1×10^{-4}	5.62×10^{0}	4.30×10^{0}	3.46×10^{0}	2.50×10^{0}	1.94×10^{0}	1.56×10^{0}	1.09×10^{0}	6.97×10^{-1}	5.08×10^{-1}
2×10^{-4}									
3.5×10^{-4}									
6×10^{-4}									
1×10^{-3}								6.97×10^{-1}	5.08×10^{-1}
2×10^{-3}								6.97×10^{-1}	5.09×10^{-1}
3.5×10^{-3}								6.98×10^{-1}	5.10×10^{-1}
6×10^{-3}								7.00×10^{-1}	5.12×10^{-1}
1×10^{-2}								7.03×10^{-1}	5.16×10^{-1}
2×10^{-2}						1.56×10^{0}	1.09×10^{0}	7.10×10^{-1}	5.24×10^{-1}
3.5×10^{-2}					1.94×10^{0}	1.56×10^{0}	1.10×10^{0}	7.20×10^{-1}	5.37×10^{-1}
6×10^{-2}				2.50×10^{0}	1.95×10^{0}	1.57×10^{0}	1.11×10^{0}	7.37×10^{-1}	5.57×10^{-1}
1×10^{-1}				2.51×10^{0}	1.96×10^{0}	1.58×10^{0}	1.13×10^{0}	7.63×10^{-1}	5.89×10^{-1}
2×10^{-1}	5.62×10^{0}	4.30×10^{0}	3.46×10^{0}	2.52×10^{0}	1.98×10^{0}	1.61×10^{0}	1.18×10^{0}	8.29×10^{-1}	6.67×10^{-1}
3.5×10^{-1}	5.63×10^{0}	4.31×10^{0}	3.47×10^{0}	2.54×10^{0}	2.01×10^{0}	1.66×10^{0}	1.24×10^{0}	9.22×10^{-1}	7.80×10^{-1}
6×10^{-1}	5.63×10^{0}	4.31×10^{0}	3.49×10^{0}	2.57×10^{0}	2.06×10^{0}	1.73×10^{0}	1.35×10^{0}	1.07×10^{0}	9.54×10^{-1}
1×10^{0}	5.63×10^{0}	4.32×10^{0}	3.51×10^{0}	2.62×10^{0}	2.13×10^{0}	1.83×10^{0}	1.50×10^{0}	1.29×10^{0}	1.20×10^{0}
2×10^{0}	5.64×10^{0}	4.35×10^{0}	3.56×10^{0}	2.73×10^{0}	2.31×10^{0}	2.07×10^{0}	1.85×10^{0}	1.72×10^{0}	1.68×10^{0}
3.5×10^{0}	5.65×10^{0}	4.38×10^{0}	3.63×10^{0}	2.88×10^{0}	2.55×10^{0}	2.37×10^{0}	2.23×10^{0}	2.17×10^{0}	2.15×10^{0}
6×10^{0}	5.67×10^{0}	4.44×10^{0}	3.74×10^{0}	3.11×10^{0}	2.86×10^{0}	2.75×10^{0}	2.68×10^{0}	2.66×10^{0}	2.65×10^{0}
1×10^{1}	5.70×10^{0}	4.52×10^{0}	3.90×10^{0}	3.40×10^{0}	3.24×10^{0}	3.18×10^{0}	3.15×10^{0}	3.14×10^{0}	3.14×10^{0}
2×10^{1}	5.76×10^{0}	4.71×10^{0}	4.22×10^{0}	3.92×10^{0}	3.85×10^{0}	3.83×10^{0}	3.82×10^{0}	3.82×10^{0}	3.82×10^{0}
3.5×10^{1}	5.85×10^{0}	4.94×10^{0}	4.58×10^{0}	4.40×10^{0}	4.38×10^{0}	4.38×10^{0}	4.37×10^{0}	4.37×10^{0}	4.37×10^{0}
6×10^{1}	5.99×10^{0}	5.23×10^{0}	5.00×10^{0}	4.92×10^{0}	4.91×10^{0}	4.91×10^{0}	4.91×10^{0}	4.91×10^{0}	4.91×10^{0}
1×10^{2}	6.16×10^{0}	5.59×10^{0}	5.46×10^{0}	5.42×10^{0}	5.42×10^{0}	5.42×10^{0}	5.42×10^{0}	5.42×10^{0}	5.42×10^{0}

Values were obtained from (1) by setting $\sigma = 10^{-9}$.

of Type A Curves for Fully Penetrating Wells

$\beta = 0.8$	$\beta = 1.0$	$\beta = 1.5$	$\beta = 2.0$	$\beta = 2.5$	$\beta = 3.0$	$\beta = 4.0$	$\beta = 5.0$	$\beta = 6.0$	$\beta = 7.0$
1.79×10^{-2}	1.70×10^{-2}	1.53×10^{-2}	1.38×10^{-2}	1.25×10^{-2}	1.13×10^{-2}	9.33×10^{-3}	7.72×10^{-3}	6.39×10^{-3}	5.30×10^{-3}
9.15×10^{-2}	8.49×10^{-2}	7.13×10^{-2}	6.03×10^{-2}	5.11×10^{-2}	4.35×10^{-2}	3.17×10^{-2}	2.34×10^{-2}	1.74×10^{-2}	1.31×10^{-2}
1.94×10^{-1}	1.75×10^{-1}	1.36×10^{-1}	1.07×10^{-1}	8.46×10^{-2}	6.78×10^{-2}	4.45×10^{-2}	3.02×10^{-2}	2.10×10^{-2}	1.51×10^{-2}
2.96×10^{-1}	2.56×10^{-1}	1.82×10^{-1}	1.33×10^{-1}	1.01×10^{-1}	7.67×10^{-2}	4.76×10^{-2}	3.13×10^{-2}	2.14×10^{-2}	1.52×10^{-2}
3.60×10^{-1}	3.00×10^{-1}	1.99×10^{-1}	1.40×10^{-1}	1.03×10^{-1}	7.79×10^{-2}	4.78×10^{-2}		2.15×10^{-2}	
3.91×10^{-1}	3.17×10^{-1}	2.03×10^{-1}	1.41×10^{-1}						
3.94×10^{-1}									
3.94×10^{-1}	3.17×10^{-1}	2.03×10^{-1}	1.41×10^{-1}	1.03×10^{-1}	7.79×10^{-2}	4.78×10^{-2}	3.13×10^{-2}	2.15×10^{-2}	1.52×10^{-2}

$$T = c_3(Q/\Delta s_E) \qquad (18)$$

where c_3 is the same as it is in (15), and this value of T should be approximately equal to that previously obtained from the late drawdown data. The elastic storage coefficient is obtained from

$$S = c_4(Tt_E/r^2) \qquad (19)$$

where c_4 is the same as it is in (16).

4. Step 4 is identical to step 3 of the type curve method, with the exception that σ can also be calculated from

$$\sigma = t_E/t_L \qquad (20)$$

This equation follows directly from (16) and (19).

The semilogarithmic approach is faster and simpler to use than the type curve method. In addition, the theoretical curve required for this method (Figure 3) can be obtained with far less computer time than the type curves shown in Figure 1.

Recovery tests. The analytical model from which (1) has been derived assumes that the delayed response process can be simulated without recourse to unsaturated flow theory merely by treating the unconfined aquifer as a compressible system and the phreatic surface as a moving material boundary. This means that hysteresis, which often characterizes the relationship between pressure head and moisture content in unsaturated soils, does not play any part in our model and thus the delayed response process is fully reversible. In other words, (1) is applicable irrespective of whether the water table is declining or rising.

Hence one can use recovery test data from the pumping well or from the observation wells to determine aquifer transmissivity. Let t be the time since pumping started and let

of Type B Curves for Fully Penetrating Wells

$\beta = 0.8$	$\beta = 1.0$	$\beta = 1.5$	$\beta = 2.0$	$\beta = 2.5$	$\beta = 3.0$	$\beta = 4.0$	$\beta = 5.0$	$\beta = 6.0$	$\beta = 7.0$
3.95×10^{-1}	3.18×10^{-1}	2.04×10^{-1}	1.42×10^{-1}	1.03×10^{-1}	7.80×10^{-2}	4.79×10^{-2}	3.14×10^{-2}	2.15×10^{-2}	1.53×10^{-2}
					7.81×10^{-2}	4.80×10^{-2}	3.15×10^{-2}	2.16×10^{-2}	1.53×10^{-2}
				1.03×10^{-1}	7.83×10^{-2}	4.81×10^{-2}	3.16×10^{-2}	2.17×10^{-2}	1.54×10^{-2}
				1.04×10^{-1}	7.85×10^{-2}	4.84×10^{-2}	3.18×10^{-2}	2.19×10^{-2}	1.56×10^{-2}
3.95×10^{-1}	3.18×10^{-1}	2.04×10^{-1}	1.42×10^{-1}	1.04×10^{-1}	7.89×10^{-2}	4.78×10^{-2}	3.21×10^{-2}	2.21×10^{-2}	1.58×10^{-2}
3.96×10^{-1}	3.19×10^{-1}	2.05×10^{-1}	1.43×10^{-1}	1.05×10^{-1}	7.99×10^{-2}	4.96×10^{-2}	3.29×10^{-2}	2.28×10^{-2}	1.64×10^{-2}
3.97×10^{-1}	3.21×10^{-1}	2.07×10^{-1}	1.45×10^{-1}	1.07×10^{-1}	8.14×10^{-2}	5.09×10^{-2}	3.41×10^{-2}	2.39×10^{-2}	1.73×10^{-2}
3.99×10^{-1}	3.23×10^{-1}	2.09×10^{-1}	1.47×10^{-1}	1.09×10^{-1}	8.38×10^{-2}	5.32×10^{-2}	3.61×10^{-2}	2.57×10^{-2}	1.89×10^{-2}
4.03×10^{-1}	3.27×10^{-1}	2.13×10^{-1}	1.52×10^{-1}	1.13×10^{-1}	8.79×10^{-2}	5.68×10^{-2}	3.93×10^{-2}	2.86×10^{-2}	2.15×10^{-2}
4.12×10^{-1}	3.37×10^{-1}	2.24×10^{-1}	1.62×10^{-1}	1.24×10^{-1}	9.80×10^{-2}	6.61×10^{-2}	4.78×10^{-2}	3.62×10^{-2}	2.84×10^{-2}
4.25×10^{-1}	3.50×10^{-1}	2.39×10^{-1}	1.78×10^{-1}	1.39×10^{-1}	1.13×10^{-1}	8.06×10^{-2}	6.12×10^{-2}	4.86×10^{-2}	3.98×10^{-2}
4.47×10^{-1}	3.74×10^{-1}	2.65×10^{-1}	2.05×10^{-1}	1.66×10^{-1}	1.40×10^{-1}	1.06×10^{-1}	8.53×10^{-2}	7.14×10^{-2}	6.14×10^{-2}
4.83×10^{-1}	4.12×10^{-1}	3.07×10^{-1}	2.48×10^{-1}	2.10×10^{-1}	1.84×10^{-1}	1.49×10^{-1}	1.28×10^{-1}	1.13×10^{-1}	1.02×10^{-1}
5.71×10^{-1}	5.06×10^{-1}	4.10×10^{-1}	3.57×10^{-1}	3.23×10^{-1}	2.98×10^{-1}	2.66×10^{-1}	2.45×10^{-1}	2.31×10^{-1}	2.20×10^{-1}
6.97×10^{-1}	6.42×10^{-1}	5.62×10^{-1}	5.17×10^{-1}	4.89×10^{-1}	4.70×10^{-1}	4.45×10^{-1}	4.30×10^{-1}	4.19×10^{-1}	4.11×10^{-1}
8.89×10^{-1}	8.50×10^{-1}	7.92×10^{-1}	7.63×10^{-1}	7.45×10^{-1}	7.33×10^{-1}	7.18×10^{-1}	7.09×10^{-1}	7.03×10^{-1}	6.99×10^{-1}
1.16×10^{0}	1.13×10^{0}	1.10×10^{0}	1.08×10^{0}	1.07×10^{0}	1.07×10^{0}	1.06×10^{0}	1.06×10^{0}	1.05×10^{0}	1.05×10^{0}
1.66×10^{0}	1.65×10^{0}	1.64×10^{0}	1.63×10^{0}	1.63×10^{0}	1.63×10^{0}	1.63×10^{0}	1.63×10^{0}	1.63×10^{0}	1.63×10^{0}
2.15×10^{0}	2.14×10^{0}	2.14×10^{0}	2.14×10^{0}	2.14×10^{0}	2.14×10^{0}	2.14×10^{0}	2.14×10^{0}	2.14×10^{0}	2.14×10^{0}
2.65×10^{0}	2.65×10^{0}	2.65×10^{0}	2.64×10^{0}	2.64×10^{0}	2.64×10^{0}	2.64×10^{0}	2.64×10^{0}	2.64×10^{0}	2.64×10^{0}
3.14×10^{0}	3.14×10^{0}	3.14×10^{0}	3.14×10^{0}	3.14×10^{0}	3.14×10^{0}	3.14×10^{0}	3.14×10^{0}	3.14×10^{0}	3.14×10^{0}
3.82×10^{0}	3.82×10^{0}	3.82×10^{0}	3.82×10^{0}	3.82×10^{0}	3.82×10^{0}	3.82×10^{0}	3.82×10^{0}	3.82×10^{0}	3.82×10^{0}
4.37×10^{0}	4.37×10^{0}	4.37×10^{0}	4.37×10^{0}	4.37×10^{0}	4.37×10^{0}	4.37×10^{0}	4.37×10^{0}	4.37×10^{0}	4.37×10^{0}
4.91×10^{0}	4.91×10^{0}	4.91×10^{0}	4.91×10^{0}	4.91×10^{0}	4.91×10^{0}	4.91×10^{0}	4.91×10^{0}	4.91×10^{0}	4.91×10^{0}
5.42×10^{0}	5.42×10^{0}	5.42×10^{0}	5.42×10^{0}	5.42×10^{0}	5.42×10^{0}	5.42×10^{0}	5.42×10^{0}	5.42×10^{0}	5.42×10^{0}

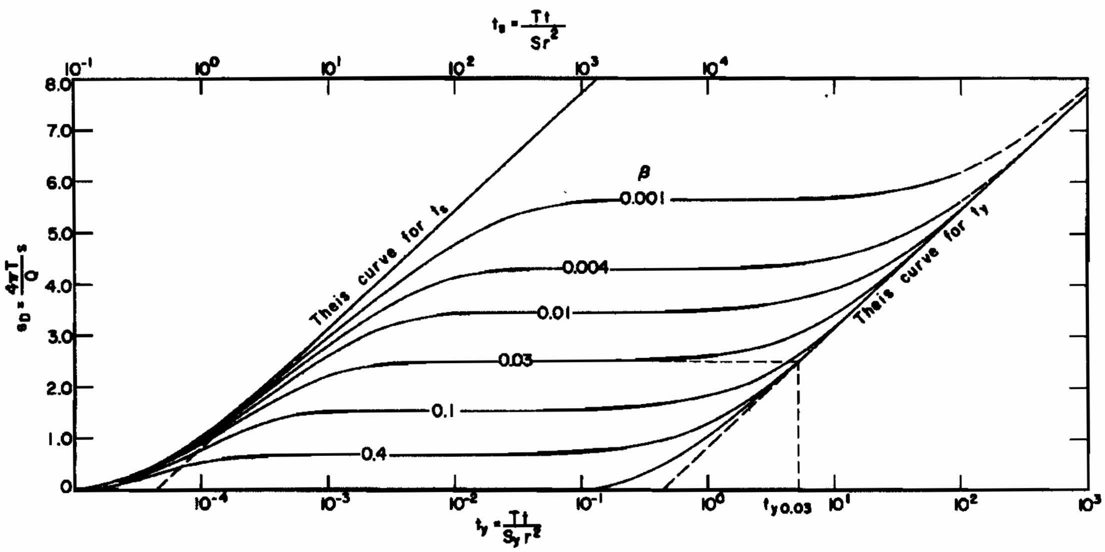

Fig. 2. Semilogarithmic relationship between dimensionless drawdown and dimensionless time in fully penetrating wells.

t_r be the time since the pump was shut off and recovery began. By plotting the residual drawdown on semilogarithmic paper versus t/t_r one finds that at large values of t_r (i.e., at small values of t/t_r) these data tend to fall on a straight line. If Δs_L is the residual drawdown corresponding to a tenfold increase in t/t_r along this straight line, then T can be calculated directly from (15).

When the pump is shut off, the water table does not respond instantly, and water first enters into storage by the expansion of the aquifer material and the compression of the water. Thus the recovery data from a given well will not fall on a straight line as long as the effect of elastic storage, created by the cessation of pumping, has not dissipated at this well.

Applicability of Jacob's correction scheme. Equation (1) was derived by assuming that the decline of the water table remains small in comparison to the saturated thickness of the aquifer. For cases where this is not so, *Jacob* [1944] recommended that prior to analyzing the pumping test data the drawdowns be corrected according to

$$s_c = s - (s^2/2b) \tag{21}$$

Prickett [1965, p. 8] advocated the use of (21) in connection with Boulton's delay yield theory.

Equation (21) was derived by adopting the Dupuit assumptions and in particular by assuming that the drawdowns along any vertical are always equal to the drawdown of the water table s_{WT}. The flux across a cylindrical surface of radius r centered around the pumping well is then given by Darcy's law as

$$q = 2\pi r K_r(b - s_{WT}) \frac{\partial(b - s_{WT})}{\partial r} = \pi r K_r \frac{\partial(b - s)^2}{\partial r} \frac{2b}{2b}$$

$$= -2\pi r b K_r \frac{\partial(s - s^2/2b)}{\partial r} = -2\pi r b K_r \frac{\partial s_c}{\partial r} \tag{22}$$

This shows that when the Dupuit assumptions hold, the equations that govern radial flow in a confined aquifer are directly applicable to unconfined aquifers provided that s is replaced by s_c.

From Figure 4 of *Neuman* [1972] it is evident that the Dupuit assumptions do not hold in an unconfined aquifer with delayed gravity response as long as the drawdown data do not fall on the late Theis curve. This means that Jacob's correction scheme is strictly applicable only to the late drawdown data and is not applicable to the early and intermediate data. It is therefore recommended that (21) be used only in the determination of T and S_y from the late drawdown data and not in the determination of β, T, and S from the early and intermediate data.

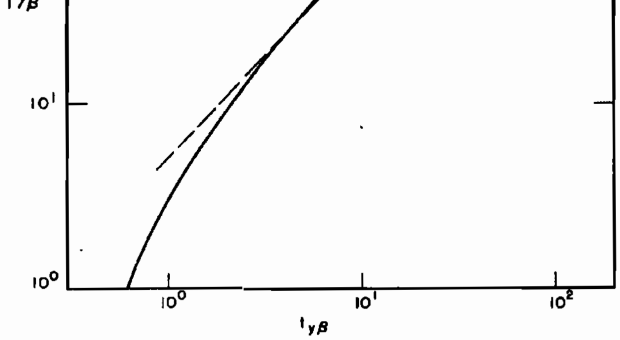

Fig. 3. Values of $1/\beta$ versus $t_{y\beta}$ for fully penetrating wells.

Applicability of distance-drawdown analyses. As was mentioned in the introduction, *Boulton* [1970] advocated the use of distance-drawdown analyses for the determination of transmissivities and storage coefficients as a supplement to the time-drawdown method. He based his recommendation on an observation made by *Wenzel* [1942], which Boulton claimed could be supported by theoretical examples, that '. . . the effect of delayed yield on distance-drawdown curves, though appreciable at large r, is unimportant compared with the effect on time-drawdown curves' [*Boulton*, 1970, p. 371].

To check whether this is correct, we first note that $\beta t_y = (TK_D t/S_y b^2)$, i.e., if βt_y is kept constant for a given aquifer, t is also constant. One can therefore develop dimensionless distance-drawdown curves from the data shown in Figure 1 merely by plotting all s_D values for which βt_y is constant against the corresponding values of t_y (since t is constant for a given value of βt_y, t_y is now directly proportional to $1/r^2$). Figure 4 shows two such distance-drawdown curves as obtained from the type B curves in Figure 1 for $\beta t_y = 4 \times 10^{-3}$ and $\beta t_y = 4 \times 10^{-1}$. The shape of these curves appears to be similar to that of the Theis curve but substantially different from the shape of the curves in Figure 1. Thus when field drawdown data from different observation wells are plotted on logarithmic paper versus $1/r^2$ at a given value of t, one may get the false impression (as has probably happened to Wenzel) that they are not affected by delayed yield and that they should be fitted to the Theis curve. Figure 4 demonstrates that when βt_y is small (i.e., when t is not large enough), analyzing the field data by matching them with the Theis solution may lead to gross errors in the evaluation of transmissivities as well as storage coefficients.

We are thus led to conclude that one should be cautious in applying distance-drawdown analyses to pumping test data from unconfined aquifers even if these data appear to fall on the Theis curve. The hydrologist will do well to heed *Prickett*'s [1965, p. 9] advice:

> The effects of delayed gravity drainage must dissipate in all observation wells before using distance-drawdown data to compute the hydraulic properties of the aquifer. During the time when delayed gravity drainage is influencing drawdown in observation wells . . . an analysis of distance-drawdown data . . . will lead to erroneous results.

It is interesting to mention that Prickett explained this statement by remarking that 'the cone of depression is distorted' during the time when delayed gravity response is important. We now know that in fact the cone of depression is not distorted [*Neuman*, 1972, Figure 7] and its slope varies monotonically with r. The true cause for the errors that may arise from using a distance-drawdown analysis is the difference between the theoretical curves for βt_y = const in Figure 4 and the Theis curve.

TABLE 2. Values of $1/\beta$ and $t_{y\beta}$ Used in Plotting Figure 3

$1/\beta$	$t_{y\beta}$
2.50×10^{-1}	4.52×10^{-1}
1.67×10^{-1}	4.55×10^{-1}
2.00×10^{-1}	4.59×10^{-1}
2.50×10^{-1}	4.67×10^{-1}
3.33×10^{-1}	4.81×10^{-1}
4.00×10^{-1}	4.94×10^{-1}
5.00×10^{-1}	5.13×10^{-1}
6.67×10^{-1}	5.45×10^{-1}
1.00×10^{0}	6.11×10^{-1}
1.25×10^{0}	6.60×10^{-1}
1.67×10^{0}	7.39×10^{-1}
2.50×10^{0}	8.93×10^{-1}
5.00×10^{0}	1.31×10^{0}
1.00×10^{1}	2.10×10^{0}
1.67×10^{1}	3.10×10^{0}
3.33×10^{1}	5.42×10^{0}
1.00×10^{2}	1.42×10^{1}
2.50×10^{2}	3.22×10^{1}
1.00×10^{3}	1.23×10^{2}

Values were obtained from (1) by setting $\sigma = 10^{-9}$.

Relationship between Boulton's delay index and aquifer characteristics. The relationship between Boulton's semiempirical quantity α (the reciprocal of which is termed delay index) and the physical characteristics of the aquifer can be obtained in the following manner. Boulton's type curves are expressed in terms of the dimensionless parameter $(r/B) = r(\alpha S_y/T)^{1/2}$, whereas our type curves are expressed in terms of β. Considering the horizontal portions of these type curves one can plot a curve of β versus s_D and another curve of (r/B) versus s_D as shown in Figure 5. By plotting the ratio between $(r/B)^2$ and β for given s_D values versus β on semilogarithmic paper one obtains the set of points shown in Figure 6. Linear regression yields the straight line

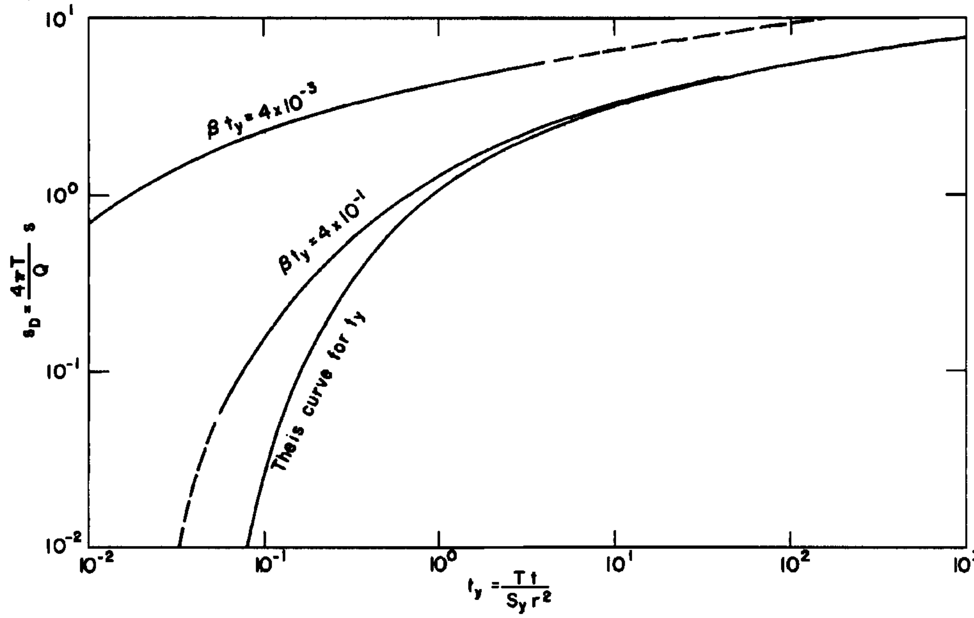

Fig. 4. Dimensionless distance-drawdown curves for fully penetrating wells at various values of βt_y = const.

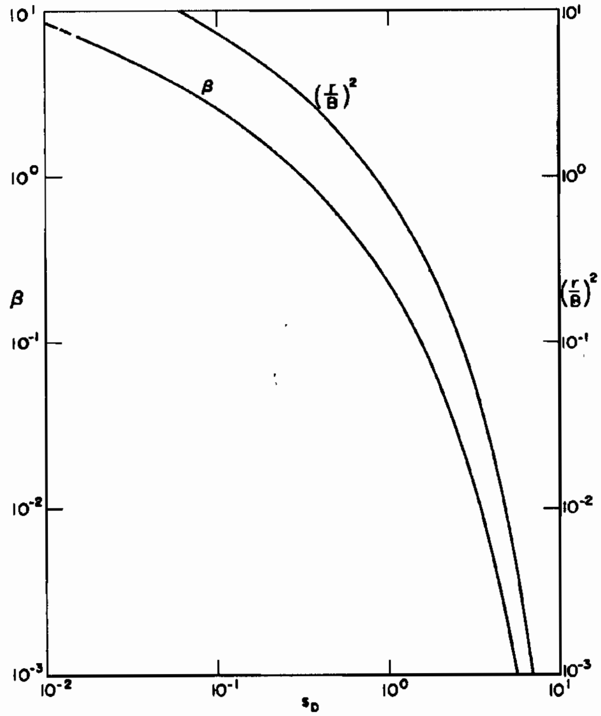

Fig. 5. Values of β and (r/B) versus s_D corresponding to horizontal portion of type curves for fully penetrating wells.

$$\frac{(r/B)^2}{\beta} = 3.063 - 0.567 \log \beta \qquad (23)$$

with a very high correlation coefficient, $\rho^2 = 0.99$. The slight scatter of the points around the straight line in Figure 6 may be due partly to the need to extrapolate the values of (r/B) from the short tables given by *Boulton* [1963, pp. 480–481] and partly to interpolation errors.

From the definitions of (r/B) and β it follows that (23) can also be written as

$$\alpha = \frac{K_z}{S_y b}\left[3.063 - 0.567 \log\left(\frac{K_D r^2}{b^2}\right)\right] \qquad (24)$$

This indicates that in a given homogeneous aquifer, α decreases in direct proportion to log r, thereby contradicting Boulton's theory in which α is assumed to be a characteristic constant of the aquifer.

Streltsova [1972, equation 3*a* and appendix] used a finite difference approximation for the delayed response process that led her to conclude that in a homogeneous aquifer $\alpha = 3K_z/[S_y(b - s_{WT})]$. Since s_{WT} is a function of r and t, it follows from this approximate relationship that α must decrease with r and increase with t. Equation (24) supports the conclusion that α decreases with r, but it contradicts the conclusion that α varies with time.

Equation (24) can be interpreted to mean that in a given aquifer the effect of delayed gravity drainage decreases linearly with the logarithm of the radial distance from the pumping well. In other words, the influence of elastic storage becomes less important as the radial distance increases, and this interpretation is consistent with what has been previously pointed out by *Neuman* [1972, Figure 6].

It is interesting to note that (23) enables one to reevaluate the results of pumping tests that have been previously obtained with the aid of Boulton's theory in light of the new theory developed by the writer, without the need to reexamine the original drawdown data. Since the values of T, S_y, and S are determined from the early and late drawdown data that fit the early and the late Theis curves, respectively, the results obtained with the aid of Boulton's theory will be practically identical to those obtained with the methods described in this paper. The difference between Boulton's theory and ours (as far as completely penetrating wells are concerned) is that the former only enables one to calculate α, whereas the latter enables one to determine the degree of anisotropy of the aquifer K_D as well as its horizontal and vertical permeabilities K_r and K_z. By knowing the values of T, S_y, and α as calculated on the basis of Boulton's theory for a given value of r, one first determines the corresponding value of $(r/B) = r(\alpha S_y/T)^{1/2}$.

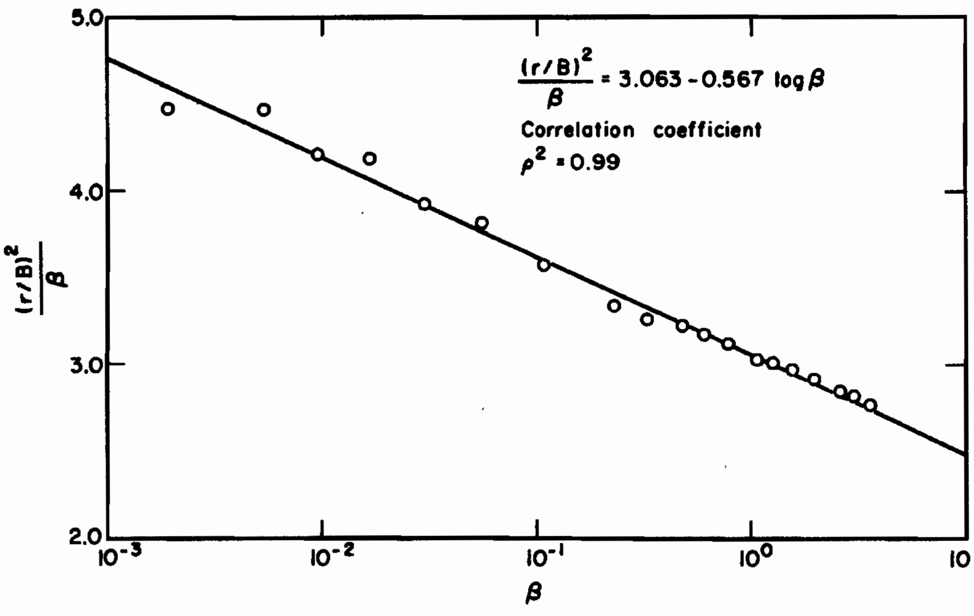

Fig. 6. Semilogarithmic relationship between $(r/B)^2/\beta$ and β for fully penetrating wells.

The value of β can then be obtained directly from (23) by using the iterative Newton-Raphson method

$$\beta^{\nu} = \beta^{\nu-1} - \frac{f(\beta)^{\nu-1}}{f'(\beta)^{\nu-1}} \tag{25}$$

where

$$f(\beta)^{\nu} = \frac{\beta^{\nu}}{(r/B)^2}(3.063 - 0.567 \log \beta) - 1$$

$$f'(\beta)^{\nu} = \left.\frac{\partial f(\beta)}{\partial \beta}\right|_{\beta=\beta^{\nu}} = \frac{1}{(r/B)^2}\left(3.063 - 0.567 \log \beta - \frac{0.567}{2.3026}\right)$$

and ν is the number of iterations. Experience with (25) indicates that a good initial estimate for β^0 is obtained by setting $\log \beta$ in (23) equal to zero, so that $\beta^0 = (r/B)^2/3.063$. Usually, not more than three iterations are required for an accuracy of $|\beta^{\nu} - \beta^{\nu-1}| \leq 0.01\beta^{\nu}$.

Once β has been determined in this manner, all the unknown aquifer parameters can be calculated by step 4 of the type curve method described earlier in the text.

Partial Penetration

When the pumping well or the observation well is perforated only through a portion of the saturated thickness of the aquifer, (1) is not applicable, and therefore the curves in Figures 1 and 3 cannot be used to analyze field data. In this case the drawdown at any point in the aquifer is given by [*Neuman*, 1974, equations 17 and 20]

$$s(r, z, t) = \frac{Q}{4\pi T}\int_0^{\infty} 4y\, J_0(y\beta^{1/2})\left[u_0(y) + \sum_{n=1}^{\infty} u_n(y)\right] dy \tag{26}$$

where

$$u_0(y) = \frac{\{1 - \exp[-t_s\beta(y^2 - \gamma_0^2)]\}\cosh(\gamma_0 z_D)}{\{y^2 + (1+\sigma)\gamma_0^2 - (y^2 - \gamma_0^2)^2/\sigma\}\cosh(\gamma_0)} \cdot \frac{\sinh[\gamma_0(1 - d_D)] - \sinh[\gamma_0(1 - l_D)]}{(l_D - d_D)\sinh(\gamma_0)} \tag{27}$$

$$u_n(y) = \frac{\{1 - \exp[-t_s\beta(y^2 + \gamma_n^2)]\}\cos(\gamma_n z_D)}{\{y^2 - (1+\sigma)\gamma_n^2 - (y^2 + \gamma_n^2)^2/\sigma\}\cos(\gamma_n)} \cdot \frac{\sin[\gamma_n(1 - d_D)] - \sin[\gamma_n(1 - l_D)]}{(l_D - d_D)\sin(\gamma_n)} \tag{28}$$

and the terms γ_0 and γ_n are the roots of (4) and (5), respectively.

The drawdown recorded in an observation well that is perforated between the elevations z_1 and z_2, where $z_1 < z_2$, is simply the average over that vertical distance. This drawdown can be calculated directly from (26) by merely redefining $u_0(y)$ and $u_n(y)$ according to [*Neuman*, 1974, equations 20–22]

$$u_0(y) = \frac{\{1 - \exp[-t_s\beta(y^2 - \gamma_0^2)]\}[\sinh(\gamma_0 z_{2D}) - \sinh(\gamma_0 z_{1D})]\{\sinh[\gamma_0(1 - d_D)] - \sinh[\gamma_0(1 - l_D)]\}}{\{y^2 + (1+\sigma)\gamma_0^2 - (y^2 - \gamma_0^2)^2/\sigma\}\cosh(\gamma_0)\cdot(z_{2D} - z_{1D})\gamma_0(l_D - d_D)\sinh(\gamma_0)} \tag{29}$$

$$u_n(y) = \frac{\{1 - \exp[-t_s\beta(y^2 + \gamma_n^2)]\}[\sin(\gamma_n z_{2D}) - \sin(\gamma_n z_{1D})]\{\sin[\gamma_n(1 - d_D)] - \sin[\gamma_n(1 - l_D)]\}}{\{y^2 - (1+\sigma)\gamma_n^2 - (y^2 + \gamma_n^2)^2/\sigma\}\cos(\gamma_n)\cdot(z_{2D} - z_{1D})\gamma_n(l_D - d_D)\sin(\gamma_n)} \tag{30}$$

Equations (26)–(28) are expressed in terms of six independent dimensionless parameters σ, β, l_D, d_D, z_D, and t_s or t_y (recall that $t_y = \sigma t_s$). Equations (26), (29), and (30) are expressed in terms of seven independent dimensionless parameters σ, β, l_D, d_D, z_{1D}, z_{2D}, and t_s or t_y. This large number of dimensionless parameters makes it practically impossible to construct a sufficient number of type curves to cover the entire range of values necessary for field application. We mentioned earlier that for a set of type curves to be useful, they should be expressed in terms of not more than two independent dimensionless parameters.

In the case of complete penetration the number of independent parameters was reduced from three to two by letting σ approach zero. The same can be done in the case of partial penetration provided that the geometric factors l_D, d_D, and z_D, or l_D, d_D, z_{1D}, and z_{2D}, are known. This procedure, however, requires that a special set of theoretical curves (such as those in Figure 1 or Figure 3) be developed for each observation well in the field, a task that is easily accomplished at the expense of a few minutes of computer time by using the program developed for this purpose by the author. As was mentioned earlier, this program, complete with users' instructions, is available from the author on request.

By letting σ in (26) approach zero, the resulting type B curves will be identical to those obtainable from *Dagan*'s [1967*a*, *b*] solution for partial penetration in an incompressible unconfined aquifer [*Neuman*, 1974, Figure 2]. *Dagan* [1967*b*] published tables of his solution for a limited number of the parameters l_D, d_D, and z_D, and the hydrologist may find these tables useful in constructing type B curves for certain field situations.

More recently *Streltsova* [1974] developed asymptotic solutions that correspond to the case in which σ in (26) approaches zero. One of her solutions yields type A curves, whereas the other is equivalent to Dagan's solution and therefore yields type B curves. Streltsova tabulated her solution for a limited number of the parameters l_D, d_D, z_{1D}, and z_{2D}, and her results may be useful in constructing type A and type B curves for situations that may sometimes arise in the field.

Another approach is to design the pumping test a priori so as to minimize the effect of partial penetration on the drawdown in the observation wells. *Neuman* [1974] showed that the effect of partial penetration on the drawdown in an unconfined aquifer decreases with radial distance from the pumping well and with the ratio $K_D = K_z/K_r$. At distances greater than $r = b/K_D^{1/2}$ this effect disappears completely when time exceeds $t = 10S_y r^2/T$ and the drawdown data follow the late Theis curve in terms of t_y. Thus if the observation well is located far from the pumping well, the late drawdown data may eventually be used to determine T and S_y by a conventional method. The early and intermediate data, however, cannot be used to determine additional aquifer parameters unless a special set of theoretical curves has been developed in the manner described earlier.

It is important to emphasize that just as in the case of the distance-drawdown analyses previously discussed, one may sometimes get the false impression that his field data are following the Theis curve, whereas in reality they fall on another curve having a similar shape. Therefore, the Theis curve should not be used to analyze late field data without having first verified that the effect of partial penetration has actually dissipated at r. One way of doing this is to install two piezometers at the same radial distance r, one at a shallow depth beneath the water table and the other at a substantially greater depth. By plotting the drawdown from both piezometers on a single sheet of logarithmic paper one will obtain two curves which tend to merge at large values of t. When the distance between these two curves becomes very small one has an indication that from a practical standpoint no vertical flow is taking place and the effect of partial penetration is thus nil.

Neuman [1974] also showed that the influence of partial penetration on the early and late drawdown data can be minimized by perforating the observation well throughout the entire saturated depth of the aquifer. In such a case, the drawdown at distances exceeding $r = b/K_D^{1/2}$ will follow the late Theis curve at times greater than $t = S_y r^2/T$, and the drawdown at distances less than $r = 0.03\, b/K_D^{1/2}$ will follow the early Theis curve at times less than $t = Sr^2/T$. Thus if a fully penetrating observation well is located far from the pumping well, its late drawdown data can be used to determine T and S_y by conventional methods. If, on the other hand, the observation well is situated at a sufficiently small distance from the pumping well, its early drawdown data may enable one to determine T and S. Here again the intermediate data from both wells are of no use in determining additional aquifer parameters unless one is able to develop theoretical curves that fit the particular situation at hand.

ANALYSIS OF DATA FROM SAINT PARDON DE CONQUES

Description of setup. Our first example concerns data from a pumping test performed in 1965 by the French Bureau de Recherches Géologiques et Minières at Saint Pardon de Conques, located in the Vallée de la Garonne, Gironde, France. An analysis of these data on the basis of Boulton's theory was described in detail by *Bonnet et al.* [1970].

The aquifer material consists of medium-grained sand with gravel in the deeper part and a clayey matrix at shallow depths. The aquifer is underlain by marls having a relatively low permeability. The bottom of the aquifer is located at a depth of 13.75 m, and the water table was initially at a depth of 5.51 m, so that $b = 8.24$ m.

The pumping well is perforated within the depth interval 7–13.75 m and has a diameter of 0.32 m. The duration of the pumping test was 48 h 50 min at a rate that oscillated between 51 and 54.6 m^3/h and averaged about 53 m^3/h. Drawdowns were monitored at radial distances of 10 and 30 m from the pumping well.

In the following analysis it is assumed that owing to the large coefficient of penetration of the pumping well, the effect of partial penetration can be neglected at $r = 10$ m and $r = 30$ m. In addition, owing to a lack of technical data about the observation wells, the latter are assumed to be perforated throughout the entire thickness of the aquifer. As will be seen below, the results of the analysis appear to be quite consistent, an indication that the influence of partial penetration on the data is sufficiently small to be neglected.

Type curve method. The variation of drawdown with time in the pumping well and at $r = 10$ m and $r = 30$ m is shown by the open circles in Figure 7. The solid lines are traces of the type curves that appeared to give the best visual fit with the data, and the open squares are the corresponding match points.

The coordinates of the match point corresponding to $r = 10$ m and the type B curve for $\beta = 0.01$ are $s^* = 0.06$ m, $s_D^* = 1$, $t^* = 200$ s = 0.0556 h, and $t_y^* = 1$. Thus according to (6) and (7), respectively, we obtain

$$T = (0.0796)\,\frac{(53)(1)}{(0.06)} = 70.3 \text{ m}^2/\text{h}$$

$$S_y = \frac{(70.3)(0.0556)}{(10)^2(1)} = 3.9 \times 10^{-2}$$

It is interesting to note that by using Boulton's theory, *Bonnet et al.* [1970] obtained by the same procedure $T = 68.0$ m^2/h and $S_y = 4.5 \times 10^{-2}$.

The coordinates of the match point corresponding to $r = 10$

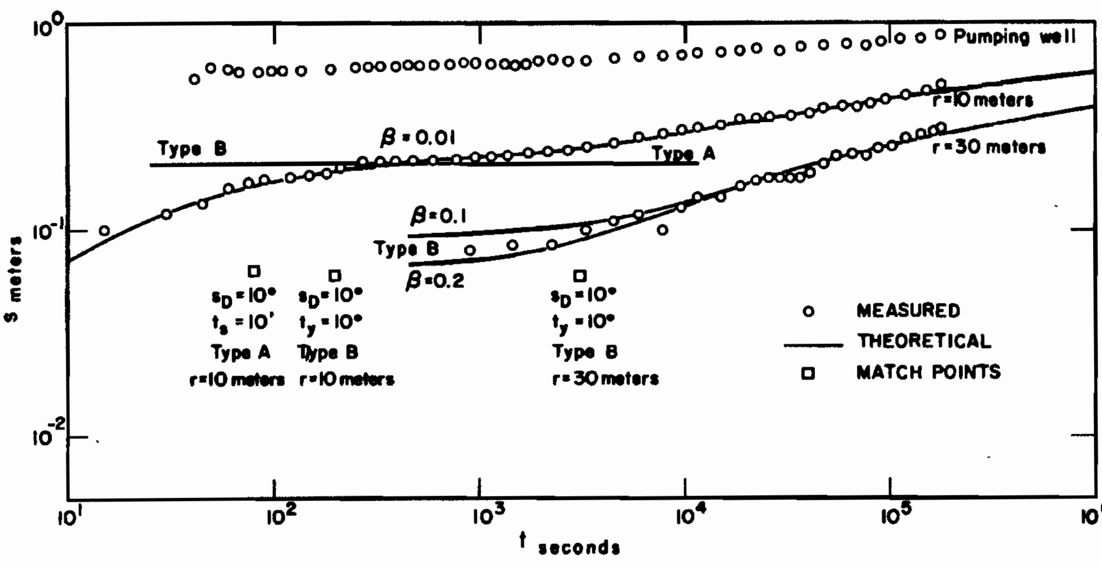

Fig. 7. Logarithmic plot of drawdown versus time at Saint Pardon de Conques.

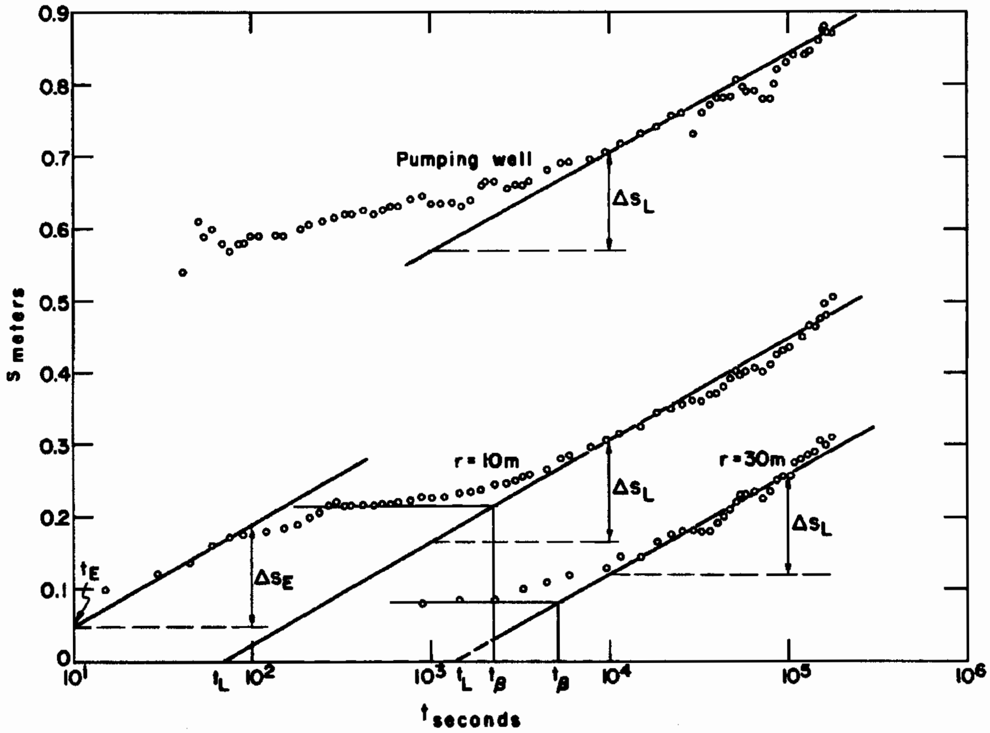

Fig. 8. Semilogarithmic plot of drawdown versus time at Saint Pardon de Conques.

m and the type A curve for $\beta = 0.01$ are $s^* = 0.064$ m, $s_D^* = 1$, $t^* = 80$ s $= 0.022$ h, and $t_s^* = 10$. By using (6) and (8) one therefore calculates

$$T = (0.0796)\frac{(53)(1)}{(0.064)} = 65.9 \text{ m}^2/\text{h}$$

$$S = \frac{(65.9)(0.022)}{(10)^2(10)} = 1.45 \times 10^{-3}$$

Bonnet et al. obtained from Boulton's theory $T = 69.0$ m²/h and $S = 1.5 \times 10^{-3}$. Since the late data give a better fit with the type curves than the early data, the results from the late data appear to be more reliable, and we shall therefore adopt the value $T = 70.3$ m²/h in the calculation below.

Having determined β, T, S_y, and S, one can now obtain all the remaining aquifer parameters from (8)–(12b):

$$K_r = \frac{(70.3)}{(8.24)} = 8.53 \text{ m/h}$$

$$K_D = \frac{(0.01)(8.24)^2}{(10)^2} = 6.79 \times 10^{-3}$$

$$K_z = (6.79 \times 10^{-3})(8.53) = 5.79 \times 10^{-2} \text{ m/h}$$

$$\sigma = \frac{(1.45 \times 10^{-3})}{(3.9 \times 10^{-2})} = 3.72 \times 10^{-2}$$

$$S_s = \frac{(1.45 \times 10^{-3})}{(8.24)} = 1.76 \times 10^{-4} \text{ m}^{-1}$$

The data at $r = 30$ m appear to fit a type B curve for $\beta \approx 0.18$. The coordinates of the corresponding match point are $s^* = 0.06$ m, $s_D^* = 1$, $t^* = 3100$ s $= 0.86$ h, and $t_y^* = 1$. From (6) and (7) one obtains

$$T = (0.0796)\frac{(53)(1)}{(0.06)} = 70.3 \text{ m}^2/\text{h}$$

$$S_y = \frac{(70.3)(0.86)}{(30)^2(1)} = 6.72 \times 10^{-2}$$

Bonnet et al. calculated by using Boulton's theory $T = 65.0$ m²/h and $S_y = 8.0 \times 10^{-2}$.

From (9)–(11) one further gets

$$K_r = \frac{(70.3)}{(8.24)} = 8.53 \text{ m/h}$$

$$K_D = \frac{(0.18)(8.24)^2}{(30)^2} = 1.36 \times 10^{-2}$$

$$K_z = (1.36 \times 10^{-2})(8.53) = 1.16 \times 10^{-1} \text{ m/h}$$

The fact that β in the two observation wells at $r = 10$ m and $r = 30$ m does not vary as the square of these radii (see definition of β in the notation) may be due to such diverse causes as aquifer inhomogeneity, partial penetration, or lack of sensitivity of the data match to the values of β.

Semilogarithmic method. Figure 8 shows the same data as Figure 7 but on a semilogarithmic scale. At $r = 10$ m, two parallel straight lines can be fitted to the late and early data and a horizontal line to the intermediate data. These lines give $\Delta s_L = 0.137$ m, $t_L = 70$ s $= 0.0194$ h, $t_\beta = 2250$ s $= 0.625$ h, $\Delta s_E = 0.138$ m, and $t_E = 4.25$ s $= 0.00118$ h. Thus according to (15) and (16), respectively, we obtain

$$T = (0.1833)\frac{(53)}{(0.137)} = 70.8 \text{ m}^2/\text{h}$$

$$S_y = (2.246)\frac{(70.8)(0.0194)}{(10)^2} = 3.08 \times 10^{-2}$$

The results obtained by Bonnet et al. are $T = 67.0$ m²/h and $S_y = 3.5 \times 10^{-2}$.

The dimensionless time $t_{y\beta}$ is computed from T, S_y, and t_β according to (17),

$$t_{y\beta} = \frac{(70.8)(0.625)}{(3.08 \times 10^{-2})(10)^2} = 14.37$$

As this falls within the range of values for which (14) applies, one can determine β either by (14) or from Figure 3. According to (14),

$$\beta = \frac{(0.195)}{(14.37)^{1.105}} = 0.01$$

To analyze the early data we use (18) and (19), respectively:

$$T = (0.1833)\frac{(53)}{(0.138)} = 70.4 \text{ m}^2/\text{h}$$

$$S = (2.246)\frac{(70.4)(0.00118)}{(10)^2} = 1.87 \times 10^{-3}$$

By adopting the average value of transmissivity $T = 70.6$ m²/h, we can now calculate all the remaining aquifer parameters with the aid of (8)–(12*b*),

$$K_r = \frac{(70.6)}{(8.24)} = 8.57 \text{ m/h}$$

$$K_D = \frac{(0.01)(8.24)^2}{(10)^2} = 6.79 \times 10^{-3}$$

$$K_z = (6.79 \times 10^{-3})(8.57) = 5.82 \times 10^{-2}$$

$$\sigma = \frac{(1.87 \times 10^{-3})}{(3.08 \times 10^{-2})} = 6.07 \times 10^{-2}$$

$$S_s = \frac{(1.87 \times 10^{-3})}{(8.24)} = 2.27 \times 10^{-4} \text{ m}^{-1}$$

At $r = 30$ m, a straight line can be fitted to the late data and a horizontal line to the intermediate data. These lines give $\Delta s_L = 0.137$ m, $t_L = 1300$ s $= 0.361$ h, and $t_\beta = 5200$ s $= 1.444$ h. By using (15) and (16) one obtains

$$T = (0.1833)\frac{(53)}{(0.137)} = 70.8 \text{ m}^2/\text{h}$$

$$S_y = (2.246)\frac{(70.8)(0.361)}{(30)^2} = 6.38 \times 10^{-2}$$

Bonnet et al. calculated from the same data $T = 63.0$ m²/h and $S_y = 6.5 \times 10^{-2}$.

The dimensionless time $t_{y\beta}$ is computed from T, S_y, and t_β according to (17),

$$t_{y\beta} = \frac{(70.8)(1.444)}{(6.38 \times 10^{-2})(30)^2} = 1.78$$

This falls outside the range of values for which (14) is applicable, and so β must be determined from Figure 3. According to this figure, $1/\beta = 7.8$, and therefore

$$\beta = (1)/(7.8) = 0.128$$

Finally, from (9)–(11) one calculates

$$K_r = (70.8)/(8.24) = 8.59 \text{ m/h}$$

$$K_D = \frac{(0.128)(8.24)^2}{(30)^2} = 9.66 \times 10^{-3}$$

$$K_z = (9.66 \times 10^{-3})(8.59) = 8.29 \times 10^{-2} \text{ m/h}$$

A straight line can also be fitted to the late drawdown data in the pumping well, giving $\Delta s_L = 0.136$ m. This together with (15) makes it possible to determine the transmissivity,

$$T = (0.1833)\frac{(53)}{(0.136)} = 71.43 \text{ m}^2/\text{h}$$

The result of Bonnet et al. for the same data is $T = 69.0$ m²/h.

Recovery test. The residual drawdown during recovery was measured in the pumping well and at $r = 10$ m for a period of 6 h 20 min. Figure 9 shows what happens when this drawdown is plotted versus t/t_r on semilogarithmic paper. Two parallel straight lines can be fitted to the late recovery data from both wells, giving $\Delta s_L = 0.137$ m. By using (15) we obtain

$$T = (0.1833)\frac{(53)}{(0.137)} = 70.8 \text{ m}^2/\text{h}$$

Bonnet et al. obtained by the same method $T = 72.0$ m²/h.

Conclusions. By averaging all the results obtained at all three wells by both the type curve and semilogarithmic methods of analysis we conclude that the aquifer has the following hydraulic properties: $T = 70.2$ m²/h, $K_r = 8.6$ m/h, $K_D = 9.2 \times 10^{-3}$, $K_z = 7.9 \times 10^{-2}$ m/h, $S_y = 5.0 \times 10^{-2}$, $S = 1.7 \times 10^{-3}$, $\sigma = 3.3 \times 10^{-2}$, and $S_s = 2.0 \times 10^{-4}$ m^{-1}. We see that the horizontal permeability is about a hundred times greater than the vertical one, a fact that may be due to the presence of clay in the upper portion of the aquifer. Furthermore, S_s and σ are relatively large, the indication being that the compressibility of the aquifer is greater than that usually encountered in deep-seated confined aquifers composed of comparable materials. A theoretical explanation for the relatively high specific storage values in shallow unconfined aquifers has recently been proposed by G. Gambolati (unpublished manuscript, 1974).

DIRECT REINTERPRETATION OF RESULTS PREVIOUSLY OBTAINED FROM BOULTON'S THEORY

We mentioned earlier that (23) enables one to reinterpret the results of pumping tests that have been previously obtained with the aid of Boulton's theory, in light of the new theory developed by the writer, without reexamining the original drawdown data (provided that the wells are fully penetrating). Two examples of this approach are given below.

Saint Pardon de Conques. This pumping test was described in the previous section. At $r = 10$ m, *Bonnet et al.* [1970] obtained by using Boulton's theory the following average values: $T = 68.25$ m²/h, $S_y = 4.0 \times 10^{-2}$, $S = 1.5 \times 10^{-3}$, $1/\alpha = 6000$ s $= 1.667$ h, and $(r/B) = 0.2$. By using the Newton-Raphson method as described in (25), we obtain $\beta^0 = 0.013$, $\beta^1 = 0.088$, $\beta^2 = 0.0096$, $\beta^3 = 0.0095$, $\beta^4 = 0.0095$. This result is very similar to the value $\beta = 0.01$, which we have previously obtained by our theory. Thus, from (8)–(10) one can now obtain

$$K_r = \frac{(68.25)}{(8.24)} = 8.28 \text{ m/h}$$

$$K_D = \frac{(0.0095)(8.24)^2}{(10)^2} = 6.45 \times 10^{-3}$$

$$K_z = (6.45 \times 10^{-3})(8.28) = 5.34 \times 10^{-2} \text{ m/h}$$

These results are practically the same as those we calculated before by using the new theory.

At $r = 30$ m, Bonnet et al. obtained from Boulton's theory $T = 64.0$ m²/h, $S_y = 7.25 \times 10^{-2}$, $1/\alpha = 6250$ s $= 1.736$ h, and $(r/B) = 0.8$. From (25) we get $\beta^0 = 0.209$, $\beta^1 = 0.178$, $\beta^2 = 0.185$, $\beta^3 = 0.184$, $\beta^4 = 0.184$, which is again very similar to the

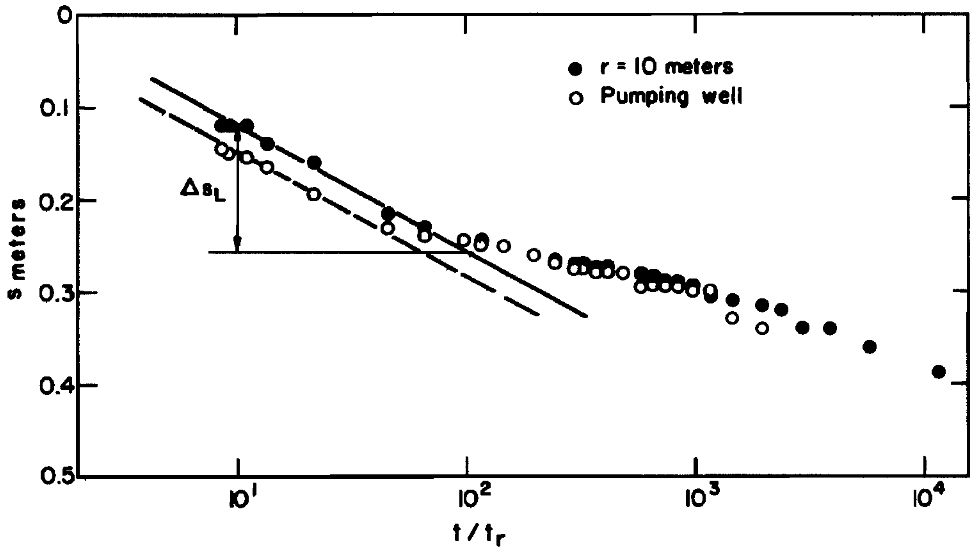

Fig. 9. Recovery data for two wells at Saint Pardon de Conques.

value β = 0.18 as previously determined by our type curve method. Thus from (8)-(10) one calculates

$$K_r = \frac{(64.0)}{(8.24)} = 7.77 \text{ m/h}$$

$$K_D = \frac{(0.184)(8.24)^2}{(30)^2} = 1.39 \times 10^{-2}$$

$$K_z = (1.39 \times 10^{-2})(7.77) = 1.08 \times 10^{-1} \text{ m/h}$$

Lawrenceville, Illinois. *Prickett* [1965, pp. 9-11] described in detail a pumping test performed in Lawrence County near Lawrenceville, Illinois. The aquifer was pumped for 24 h at a rate of 1000 gpm, its initial saturated thickness being 100 feet. The drawdown data were corrected for partial penetration by a method proposed by *Jacob* [1963]. At r = 200 ft, Prickett calculated by using Boulton's theory the following average values for the aquifer parameters: T = 272,500 gpd/ft, $S_y = 3.17 \times 10^{-2}$, $S = 2.33 \times 10^{-3}$, $1/\alpha$ = 104 min, and $(r/B) = 0.7$. By employing the Newton-Raphson method as described in (25) and assuming that the effect of partial penetration is small, we obtain $\beta^0 = 0.160, \beta^1 = 0.133, \beta^2 = 0.139, \beta^3 = 0.138, \beta^4 = 0.138$. This together with (8)-(10) gives

$$K_r = \frac{(272{,}500)}{(100)} = 2725.0 \text{ gpd/ft}^2$$

$$K_D = \frac{(0.138)(100)^2}{(200)^2} = 0.0345$$

$$K_z = (0.0345)(2725.0) = 94.0 \text{ gpd/ft}^2$$

Thus the horizontal permeability is about 30 times as large as the vertical permeability, which is reasonable in light of the local geology as described by the driller's log. More accurate results could be obtained by reexamining the original data and by taking into account the effect of partial penetration in the present analysis.

NOTATION

- b, initial saturated thickness of aquifer, L;
- c_i, constants depending on units;
- d, vertical distance between top of perforations in pumping well and initial position of water table, L;
- d_D, dimensionless d, equal to d/b;
- $J_0(x)$, zero-order Bessell function of the first kind;
- K_r, horizontal permeability, LT^{-1};
- K_z, vertical permeability, LT^{-1};
- K_D, degree of anisotropy, equal to K_z/K_r;
- l, vertical distance between bottom of perforations in pumping well and initial position of water table, L;
- l_D, dimensionless l, equal to l/b;
- Q, pumping rate, L^3T^{-1};
- r, radial distance from pumping well, L;
- r/B, Boulton's parameter, equal to $r(\alpha S_y/T)^{1/2}$;
- s, drawdown, L;
- s_c, drawdown corrected according to Jacob's procedure, equal to $s - s^2/b$, L;
- s_D, dimensionless drawdown, equal to $4\pi Ts/Q$;
- s_{WT}, drawdown of the water table, L;
- S, storage coefficient, equal to $S_s b$;
- S_s, specific (elastic) storage, L^{-1};
- S_y, specific yield;
- t, time since pumping started, T;
- t_r, time since recovery started, T;
- t_s, dimensionless time with respect to S_s, equal to Tt/Sr^2;
- t_y, dimensionless time with respect to S_y, equal to Tt/S_yr^2;
- $t_{y\beta}$, t_y corresponding to intersection of horizontal line through intermediate data with inclined line through late data in Figure 2;
- t_E, t corresponding to intersection of straight line through early drawdown data with s = 0 on semilogarithmic paper, T;
- t_L, t corresponding to intersection of straight line through late drawdown data with s = 0 on semilogarithmic paper, T;
- t_β, t corresponding to intersection of horizontal line through intermediate drawdown data with inclined line through late data on semilogarithmic paper, T;
- T, transmissivity $K_r b$, L^2T^{-1};
- z, vertical distance above bottom of aquifer, L;
- z_D, dimensionless elevation, equal to z/b;

α, reciprocal of Boulton's delay index, T^{-1};

$\beta = K_D r^2/b^2$;

Δs_E, tenfold increase in drawdown along straight line through early data on semilogarithmic paper, L;

Δs_L, tenfold increase in drawdown along straight line through late data on semilogarithmic paper, L;

$\sigma = S/S_y$.

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