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# DRAWDOWN AROUND A PARTIALLY PENETRATING WELL

By Mahdi S. Hantush[1]

## SYNOPSIS

Equations of unsteady drawdown around a well partially screened in and steadily discharging from an artesian aquifer of uniform thickness and uniform hydraulic properties are developed. The discharge is supplied by the reduction of storage through expansion of water and the concomitant compression of the aquifer. The solutions are put in forms amenable to relatively simple computation. The results are compared with that of the case of complete penetraion. Application of the theory to analysis of aquifer tests will be given in a subsequent paper.

## INTRODUCTION

Producing wells frequently do not completely penetrate the aquifer from which they are pumping. The hydraulics of such wells is therefore different rom that of wells that fully penetrate the aquifer. The problem of partial penetration has long been recognized, and approximate steady-state solutions or various field conditions have been advanced by J. Kozeny,[2] A. M. ASCE

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[1] Senior Hydrol. and Prof. of Hydr., New Mexico Inst. of Mining and Tech., Socorro, N. M. On leave from the College of Engineering, Univ. of Baghdad, Iraq.

[2] "Theorie und Berechnung der Brunnen," by J. Kozeny, Wasserkraft u. Wasserwirtschaft, Vol. 28, 1933, p. 101.

M. Muskat,[3] P. Ya. Polubarinova-Kochina,[4] and Hantush and C. E. Jacob,[5] F. ASCE. An exact theory of steady-state flow into a partially penetrating well has been developed by Don Kirkham.[6] The use of these solutions is limited by the fact that they describe the flow under the equilibrium condition only, a condition rarely attained during actual periods of well operation.

The purpose of this paper is to develop unsteady drawdown equations by taking into consideration the length and space positon of the water entry face (screened section) of both the pumped well and the observation well.

*Notation.*—The symbols in this paper are defined where they first appear. They are assembled alphabetically for convenience, in the Appendix.

## DRAWDOWN EQUATIONS

A well whose length of water entry is less than the thickness of the aquifer it penetrates is known as a partially penetrating well. Fig. 1 illustrates conditions characteristic of partially penetrating wells in a confined aquifer. The flow pattern to such wells is three-dimensional rather than the radial (two-dimensional) flow assumed to exist around fully penetrating wells. The drawdown around these wells depends, therefore, among other things, on the space position of the point of observation. Consequently, the drawdown observed in partially penetrating wells will depend on the length and the space position of the screened or perforated sections (water entry portion) of the observation wells.

## DRAWDOWN IN PIEZOMETERS

Piezometers are small diameter pipes driven into an aquifer, so that entrance of water into them is solely from the bottom. The drawdown s in a piezometer having a depth of penetration z and being at a distance r from a steadily discharging well that is screened throughout its depth of penetration l, has been obtained by the writer[7] for the case in which the pumped well is of a small radius and the aquifer is of a uniform thickness b, homogeneous, elastic, isotropic, nonleaky, and infinite in areal extent (see Fig. 1 for drawdown equations of this case). A similar analysis will yield a more general solution for the problem if the screen of the pumped well does not extend to the top of the aquifer. In the case of a nonleaky aquifer that is drained by a well of a constant discharge Q and whose screen lies between the depths l and d $(l > d)$, the

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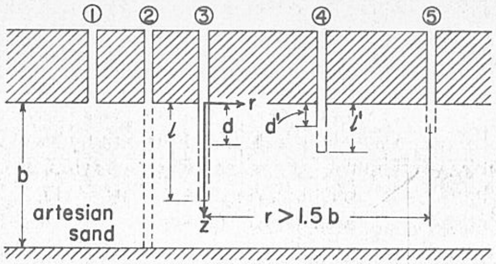
[3] "The Flow of Homogeneous Fluids Through Porous Media," by M. Muskat, McGraw-Hill Book Co., Inc., New York, N. Y., 1937; or J. W. Edwards Brothers, Inc., Ann Arbor, Mich., 1946.

[4] "Theory of Filtration of Liquids in Porous Media," by P. Ya, Polubarinova-Kochina, Advances in Applied Mechanics, Vol. 2, 1951, p. 207.

[5] "Steady Three-Dimensional Flow to a Well in a Two-Layered Aquifer," by M. S. Hantush and C. E. Jacob, Transactions, Amer. Geophysical Union, Vol. 36, 1955, p. 286.

√ [6] "Exact Theory of Flow Into a Partially Penetrating Well," by Don Kirkham, Journal of Geophysical Research, Vol. 64, 1959, p. 1317.

[7] "Nonsteady Flow to a Well Partially Penetrating an Infinite Leaky Aquifer," by M. S. Hantush, Proceedings, Iraq i Scientific Soc., 1957, pp. 10-19; also reprinted by New Mexico Inst. of Mining and Tech., Socorro, N. M.



Equations of drawdown if $d = d' = 0$

$$u = \frac{r^2 S_s}{4Kt}$$

in: ② & ⑤ $\bar{s} = \frac{Q}{4\pi K b} W(u)$

① & ③ $s = \frac{Q}{4\pi K \ell}\left[\frac{\ell}{b} W(u) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi\ell}{b} W_n\left(u, \frac{n\pi r}{b}\right)\right]$

④ $\bar{s} = \frac{Q}{4\pi K \ell}\left[\frac{\ell}{b} W(u) + \frac{2b}{\pi^2 \ell'} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi\ell}{b} \sin\frac{n\pi\ell'}{b} W\left(u, \frac{n\pi r}{b}\right)\right]$

for $b = \infty$ $s(r,z,t) = \frac{Q}{8\pi K \ell} \int_u^{\infty} \frac{e^{-y}}{y}\left[\operatorname{erf}\left(\frac{\ell - z}{r}\sqrt{y}\right) + \operatorname{erf}\left(\frac{\ell + z}{r}\sqrt{y}\right)\right] \cdot dy$

$$W(u) = \int_u^{\infty} \frac{e^{-y}}{y} dy$$

$$W\left(u, \frac{n\pi r}{b}\right) = \int_u^{\infty} \frac{dy}{y} \exp\left[-y - \frac{\left(\frac{n\pi r}{b}\right)^2}{4y}\right]$$

FIG. 1.—DIAGRAMMATIC REPRESENTATION OF WELLS PARTIALLY PENETRATING AN ARTESIAN AQUIFER

solution is given by either of the following two equations (see Fig. 1 for coordinate system):

$$s = \frac{Q}{4\pi K b}\left[W(u) + f\left(u, \frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}\right)\right] \quad \text{. . . . . . . . . . . . (1)}$$

or

$$s = \frac{Q}{8\pi K (l - d)}\left[M\left(u, \frac{l + z}{r}\right) + M\left(u, \frac{l - z}{r}\right) + f'\left(u, \frac{b}{r}, \frac{l}{r}, \frac{z}{r}\right) - M\left(u, \frac{d + z}{r}\right) - M\left(u, \frac{d - z}{r}\right) - f'\left(u, \frac{b}{r}, \frac{d}{r}, \frac{z}{r}\right)\right] \quad \text{. . . . . . . . (2a)}$$

in which

$$f = \frac{2b}{\pi(l - d)} \sum_{n=1}^{\infty} \frac{1}{n}\left(\sin\frac{n\pi l}{b} - \sin\frac{n\pi d}{b}\right) \cos\frac{n\pi z}{b} W\left(u, \frac{n\pi r}{b}\right) \quad \text{. . . (2b)}$$

and

$$f'\left(u, \frac{b}{r}, \frac{x}{r}, \frac{z}{r}\right) \sum_{n=1}^{\infty}\left[M\left(u, \frac{2nb + x + z}{r}\right) - M\left(u, \frac{2nb - x - z}{r}\right) + M\left(u, \frac{2nb + x - z}{r}\right) - M\left(u, \frac{2nb - x + z}{r}\right)\right] \quad \text{. . . (2c)}$$

and where

$$u = \frac{r^2 S_s}{4\,K\,t}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2d)$$

t is the time since pumping started, K is the hydraulic conductivity, $S_s$ is the specific storage (volume of water released from storage in a unit volume of the aquifer under a unit head decline, a dimension $L^{-1}$), and W(u,y) is[8] what is known as the well function for leaky aquifers. The latter function has been tabulated extensively.

The function $M(u, \beta)$ is defined by the following infinite integral

$$M(u, \beta) = \int_u^\infty \frac{e^{-y}}{y}\, \mathrm{erf}\,(\beta\sqrt{y})\, dy \quad \ldots\ldots\ldots \quad (3a)$$

in which erf(x) is the error function. Because erf (-x) = -erf(x), it follows that

$$M(u, -\beta) = -M(u, \beta) \quad \ldots\ldots\ldots\ldots \quad (3b)$$

The function $M(u,\beta)$ has[9] been tabulated for a sufficient range of the parameters involved and is given in Table 1. The function can be approximated with sufficient accuracy by

$$M(u, \beta) = 2\left(\sinh^{-1}\beta \;\; \frac{2}{\sqrt{\pi}}\,\beta\sqrt{u}\right), \text{ if } u < \frac{0.05}{2} < .01 \ \ldots \quad (4)$$

$$M(u, \beta) = 2\left(\sinh^{-1}\beta - \beta\,\mathrm{erf}\,(\sqrt{u})\right), \text{ if } u < \frac{.05}{\beta^2} \ \ldots\ldots\ldots \quad (5)$$

and

$$M(u, \beta) = W(u) \qquad , \text{ if } u > \frac{5}{\beta^2} \ \ldots\ldots\ldots \quad (6)$$

in which $\sinh^{-1}\beta$ is the inverse hyperbolic sine of $\beta$ and W(u) is the well function for nonleaky aquifers, or what in the mathematical literature is known as the negative exponential integral of (-u). This function is available in tabular form.[10]

*Equation of Drawdown for Relatively Small Values of Time.*—By virtue of Eq. 6, f' terms of Eq. 2a can be safely neglected if $u > 5\left[r/(2\,b - 1 - z)\right.^2$. For b equals infinity these f' terms vanish also. Thus, if either $t < \left[(2\,b - 1 - z)^2 S_s/(20\,K)\right]$; that is for relatively short period of pumping, or the aquifer is

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[8] "Preliminary Quantitative Study of the Roswell Ground-Water Reservoir," by M. S. Hantush, New Mexico Inst. of Mining and Tech., 1955 (reprinted, 1957); also "Analysis of Data from Pumping Tests in Leaky Aquifers," by M. S. Hantush, Transactions, Amer. Geophysical Union, Vol. 37, 1956, p. 702.

[9] Professional Paper 102, Research Div., New Mexico Inst. of Mining and Tech., Socorro, N. M.

[10] "Methods for Determining Permeability of Water-Bearing Materials," by L. K. Wenzel, U. S. Geol. Survey, Water-Supply Paper No. 887, 1942, p. 88, also "Hydrology," by C. O. Wisler and E. F. Brater, John Wiley and Sons, New York, N. Y., 1951.

## TABLE 1.—VALUES OF THE FUNCTION $M(u,\beta) = \int_u^\infty \frac{e^{-y}}{y} \operatorname{erf}(\beta\sqrt{y})\, dy$

| u \ β | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.2 | 1.4 | 1.6 | 1.8 | 2.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0.1997 | 0.3974 | 0.5913 | 0.7801 | 0.9624 | 1.1376 | 1.3053 | 1.4653 | 1.6177 | 1.7627 | 2.0319 | 2.2759 | 2.4979 | 2.7009 | 2.8872 |
| $10^{-6}$ | 1 | 0.1994 | 0.3969 | 0.5907 | 0.7792 | 0.9613 | 1.1363 | 1.3037 | 1.4635 | 1.6157 | 1.7605 | 2.0292 | 2.2728 | 2.4943 | 2.6968 | 2.8827 |
| | 2 | 0.1993 | 0.3967 | 0.5904 | 0.7788 | 0.9608 | 1.1357 | 1.3031 | 1.4628 | 1.6148 | 1.7595 | 2.0281 | 2.2715 | 2.4929 | 2.6951 | 2.8809 |
| | 3 | 0.1993 | 0.3966 | 0.5902 | 0.7785 | 0.9605 | 1.1353 | 1.3026 | 1.4622 | 1.6142 | 1.7588 | 2.0272 | 2.2705 | 2.4917 | 2.6938 | 2.8794 |
| | 4 | 0.1992 | 0.3965 | 0.5900 | 0.7783 | 0.9602 | 1.1349 | 1.3022 | 1.4617 | 1.6137 | 1.7582 | 2.0265 | 2.2696 | 2.4907 | 2.6927 | 2.8782 |
| | 5 | 0.1992 | 0.3964 | 0.5898 | 0.7780 | 0.9599 | 1.1346 | 1.3018 | 1.4613 | 1.6132 | 1.7577 | 2.0259 | 2.2689 | 2.4899 | 2.6918 | 2.8772 |
| | 6 | 0.1991 | 0.3963 | 0.5897 | 0.7779 | 0.9596 | 1.1343 | 1.3014 | 1.4609 | 1.6127 | 1.7572 | 2.0253 | 2.2682 | 2.4891 | 2.6909 | 2.8762 |
| | 7 | 0.1991 | 0.3962 | 0.5895 | 0.7777 | 0.9594 | 1.1341 | 1.3011 | 1.4605 | 1.6123 | 1.7568 | 2.0248 | 2.2676 | 2.4884 | 2.6901 | 2.8753 |
| | 8 | 0.1990 | 0.3961 | 0.5894 | 0.7775 | 0.9592 | 1.1338 | 1.3009 | 1.4602 | 1.6120 | 1.7563 | 2.0243 | 2.2670 | 2.4877 | 2.6894 | 2.8745 |
| | 9 | 0.1990 | 0.3960 | 0.5893 | 0.7774 | 0.9590 | 1.1336 | 1.3006 | 1.4599 | 1.6116 | 1.7560 | 2.0238 | 2.2665 | 2.4871 | 2.6887 | 2.8737 |
| $10^{-5}$ | 1 | 0.1989 | 0.3959 | 0.5892 | 0.7772 | 0.9588 | 1.1334 | 1.3003 | 1.4596 | 1.6113 | 1.7556 | 2.0234 | 2.2660 | 2.4865 | 2.6880 | 2.8730 |
| | 2 | 0.1987 | 0.3954 | 0.5883 | 0.7760 | 0.9574 | 1.1316 | 1.2983 | 1.4572 | 1.6086 | 1.7526 | 2.0198 | 2.2618 | 2.4818 | 2.6827 | 2.8671 |
| | 3 | 0.1984 | 0.3949 | 0.5876 | 0.7751 | 0.9562 | 1.1302 | 1.2967 | 1.4554 | 1.6066 | 1.7504 | 2.0171 | 2.2587 | 2.4782 | 2.6786 | 2.8625 |
| | 4 | 0.1982 | 0.3945 | 0.5871 | 0.7744 | 0.9553 | 1.1291 | 1.2953 | 1.4530 | 1.6049 | 1.7485 | 2.0148 | 2.2560 | 2.4751 | 2.6752 | 2.8587 |
| | 5 | 0.1981 | 0.3942 | 0.5866 | 0.7737 | 0.9544 | 1.1281 | 1.2941 | 1.4526 | 1.6034 | 1.7468 | 2.0128 | 2.2536 | 2.4724 | 2.6721 | 2.8553 |
| | 6 | 0.1979 | 0.3939 | 0.5861 | 0.7731 | 0.9537 | 1.1271 | 1.2931 | 1.4513 | 1.6020 | 1.7452 | 2.0110 | 2.2515 | 2.4700 | 2.6694 | 2.8523 |
| | 7 | 0.1978 | 0.3936 | 0.5857 | 0.7725 | 0.9530 | 1.1263 | 1.2921 | 1.4502 | 1.6007 | 1.7438 | 2.0093 | 2.2495 | 2.4677 | 2.6669 | 2.8495 |
| | 8 | 0.1976 | 0.3933 | 0.5853 | 0.7720 | 0.9523 | 1.1255 | 1.2912 | 1.4492 | 1.5996 | 1.7425 | 2.0077 | 2.2477 | 2.4657 | 2.6645 | 2.8469 |
| | 9 | 0.1975 | 0.3931 | 0.5849 | 0.7715 | 0.9517 | 1.1248 | 1.2903 | 1.4482 | 1.5984 | 1.7413 | 2.0062 | 2.2460 | 2.4637 | 2.6623 | 2.8444 |
| $10^{-4}$ | 1 | 0.1974 | 0.3929 | 0.5846 | 0.7710 | 0.9511 | 1.1241 | 1.2895 | 1.4473 | 1.5974 | 1.7402 | 2.0049 | 2.2444 | 2.4619 | 2.6603 | 2.8421 |
| | 2 | 0.1965 | 0.3910 | 0.5818 | 0.7673 | 0.9465 | 1.1185 | 1.2830 | 1.4398 | 1.5890 | 1.7308 | 1.9936 | 2.2313 | 2.4469 | 2.6434 | 2.8234 |
| | 3 | 0.1958 | 0.3896 | 0.5796 | 0.7644 | 0.9429 | 1.1142 | 1.2780 | 1.4341 | 1.5825 | 1.7236 | 1.9850 | 2.2212 | 2.4354 | 2.6305 | 2.8091 |
| | 4 | 0.1952 | 0.3883 | 0.5778 | 0.7620 | 0.9398 | 1.1106 | 1.2737 | 1.4292 | 1.5771 | 1.7176 | 1.9778 | 2.2128 | 2.4258 | 2.6197 | 2.7970 |
| | 5 | 0.1946 | 0.3873 | 0.5762 | 0.7599 | 0.9372 | 1.1074 | 1.2700 | 1.4250 | 1.5723 | 1.7123 | 1.9714 | 2.2053 | 2.4172 | 2.6101 | 2.7864 |
| | 6 | 0.1941 | 0.3863 | 0.5748 | 0.7580 | 0.9348 | 1.1045 | 1.2666 | 1.4211 | 1.5680 | 1.7075 | 1.9656 | 2.1986 | 2.4095 | 2.6014 | 2.7768 |
| | 7 | 0.1937 | 0.3854 | 0.5734 | 0.7562 | 0.9326 | 1.1018 | 1.2635 | 1.4176 | 1.5640 | 1.7030 | 1.9603 | 2.1924 | 2.4025 | 2.5934 | 2.7679 |
| | 8 | 0.1933 | 0.3846 | 0.5722 | 0.7545 | 0.9305 | 1.0994 | 1.2607 | 1.4143 | 1.5603 | 1.6989 | 1.9554 | 2.1866 | 2.3959 | 2.5860 | 2.7597 |
| | 9 | 0.1929 | 0.3838 | 0.5710 | 0.7530 | 0.9286 | 1.0970 | 1.2579 | 1.4112 | 1.5568 | 1.6951 | 1.9507 | 2.1812 | 2.3897 | 2.5791 | 2.7519 |
| $10^{-3}$ | 1 | 0.1925 | 0.3831 | 0.5699 | 0.7515 | 0.9267 | 1.0948 | 1.2554 | 1.4083 | 1.5535 | 1.6914 | 1.9463 | 2.1761 | 2.3838 | 2.5725 | 2.7446 |
| | 2 | 0.1896 | 0.3772 | 0.5611 | 0.7397 | 0.9120 | 1.0771 | 1.2347 | 1.3846 | 1.5270 | 1.6619 | 1.9109 | 2.1348 | 2.3367 | 2.5195 | 2.6857 |
| | 3 | 0.1873 | 0.3727 | 0.5543 | 0.7307 | 0.9007 | 1.0636 | 1.2189 | 1.3666 | 1.5066 | 1.6393 | 1.8838 | 2.1032 | 2.3006 | 2.4788 | 2.6406 |
| | 4 | 0.1854 | 0.3689 | 0.5486 | 0.7231 | 0.8912 | 1.0521 | 1.2056 | 1.3513 | 1.4895 | 1.6203 | 1.8610 | 2.0766 | 2.2702 | 2.4447 | 2.6027 |
| | 5 | 0.1837 | 0.3655 | 0.5435 | 0.7163 | 0.8828 | 1.0421 | 1.1938 | 1.3379 | 1.4744 | 1.6035 | 1.8409 | 2.0532 | 2.2434 | 2.4146 | 2.5693 |
| | 6 | 0.1822 | 0.3625 | 0.5390 | 0.7103 | 0.8752 | 1.0330 | 1.1832 | 1.3258 | 1.4608 | 1.5884 | 1.8228 | 2.0320 | 2.2193 | 2.3875 | 2.5393 |
| | 7 | 0.1808 | 0.3597 | 0.5348 | 0.7047 | 0.8682 | 1.0246 | 1.1735 | 1.3147 | 1.4483 | 1.5745 | 1.8061 | 2.0126 | 2.1972 | 2.3626 | 2.5117 |
| | 8 | 0.1795 | 0.3571 | 0.5310 | 0.6995 | 0.8618 | 1.0169 | 1.1645 | 1.3044 | 1.4367 | 1.5616 | 1.7907 | 1.9946 | 2.1766 | 2.3395 | 2.4861 |
| | 9 | 0.1783 | 0.3547 | 0.5273 | 0.6947 | 0.8557 | 1.0096 | 1.1560 | 1.2947 | 1.4258 | 1.5495 | 1.7762 | 1.9777 | 2.1573 | 2.3179 | 2.4620 |
| $10^{-2}$ | 1 | 0.1772 | 0.3524 | 0.5239 | 0.6901 | 0.8500 | 1.0027 | 1.1480 | 1.2855 | 1.4155 | 1.5381 | 1.7625 | 1.9617 | 2.1391 | 2.2975 | 2.4394 |
| | 2 | 0.1680 | 0.3340 | 0.4962 | 0.6533 | 0.8040 | 0.9476 | 1.0836 | 1.2121 | 1.3329 | 1.4464 | 1.6527 | 1.8340 | 1.9935 | 2.1342 | 2.2587 |
| | 3 | 0.1610 | 0.3200 | 0.4753 | 0.6253 | 0.7691 | 0.9057 | 1.0349 | 1.1564 | 1.2703 | 1.3770 | 1.5697 | 1.7376 | 1.8839 | 2.0116 | 2.1233 |
| | 4 | 0.1551 | 0.3083 | 0.4578 | 0.6020 | 0.7400 | 0.8708 | 0.9942 | 1.1100 | 1.2183 | 1.3193 | 1.5008 | 1.6577 | 1.7932 | 1.9103 | 2.0117 |
| | 5 | 0.1500 | 0.2981 | 0.4425 | 0.5817 | 0.7146 | 0.8404 | 0.9588 | 1.0696 | 1.1730 | 1.2691 | 1.4410 | 1.5884 | 1.7147 | 1.8229 | 1.9156 |
| | 6 | 0.1455 | 0.2890 | 0.4289 | 0.5635 | 0.6919 | 0.8132 | 0.9272 | 1.0336 | 1.1326 | 1.2243 | 1.3877 | 1.5268 | 1.6450 | 1.7454 | 1.8307 |
| | 7 | 0.1413 | 0.2807 | 0.4164 | 0.5470 | 0.6713 | 0.7885 | 0.8994 | 1.0008 | 1.0958 | 1.1837 | 1.3394 | 1.4711 | 1.5821 | 1.6756 | 1.7543 |
| | 8 | 0.1375 | 0.2731 | 0.4050 | 0.5317 | 0.6522 | 0.7658 | 0.8720 | 0.9707 | 1.0621 | 1.1464 | 1.2951 | 1.4200 | 1.5246 | 1.6120 | 1.6848 |
| | 9 | 0.1339 | 0.2660 | 0.3943 | 0.5176 | 0.6346 | 0.7447 | 0.8474 | 0.9428 | 1.0308 | 1.1118 | 1.2541 | 1.3729 | 1.4716 | 1.5534 | 1.6210 |
| $10^{-1}$ | 1 | 0.1306 | 0.2593 | 0.3844 | 0.5043 | 0.6181 | 0.7249 | 0.8245 | 0.9167 | 1.0016 | 1.0795 | 1.2159 | 1.3290 | 1.4223 | 1.4991 | 1.5619 |
| | 2 | 0.1051 | 0.2084 | 0.3081 | 0.4030 | 0.4920 | 0.5744 | 0.6500 | 0.7186 | 0.7806 | 0.8362 | 0.9297 | 1.0029 | 1.0595 | 1.1026 | 1.1352 |
| | 3 | 8.74(-2) | 0.1731 | 0.2554 | 0.3331 | 0.4053 | 0.4713 | 0.5309 | 0.5842 | 0.6313 | 0.6727 | 0.7400 | 0.7899 | 0.8261 | 0.8519 | 0.8699 |
| | 4 | 7.39(-2) | 0.1462 | 0.2153 | 0.2801 | 0.3397 | 0.3935 | 0.4415 | 0.4837 | 0.5203 | 0.5519 | 0.6015 | 0.6363 | 0.6602 | 0.6760 | 0.6863 |
| | 5 | 6.32(-2) | 0.1248 | 0.1835 | 0.2381 | 0.2878 | 0.3323 | 0.3714 | 0.4052 | 0.4341 | 0.4584 | 0.4955 | 0.5203 | 0.5362 | 0.5462 | 0.5521 |
| | 6 | 5.44(-2) | 0.1074 | 0.1575 | 0.2039 | 0.2458 | 0.2828 | 0.3149 | 0.3423 | 0.3652 | 0.3842 | 0.4122 | 0.4300 | 0.4408 | 0.4471 | 0.4506 |
| | 7 | 4.71(-2) | 9.29(-2) | 0.1360 | 0.1756 | 0.2111 | 0.2421 | 0.2686 | 0.2909 | 0.3093 | 0.3242 | 0.3455 | 0.3583 | 0.3657 | 0.3698 | 0.3719 |
| | 8 | 4.10(-2) | 8.06(-2) | 0.1179 | 0.1519 | 0.1821 | 0.2082 | 0.2302 | 0.2484 | 0.2632 | 0.2750 | 0.2913 | 0.3007 | 0.3058 | 0.3084 | 0.3096 |
| | 9 | 3.57(-2) | 7.03(-2) | 0.1026 | 0.1319 | 0.1576 | 0.1797 | 0.1980 | 0.2130 | 0.2250 | 0.2343 | 0.2468 | 0.2537 | 0.2572 | 0.2589 | 0.2597 |
| 1 | 1 | 3.13(-2) | 6.14(-2) | 8.95(-2) | 0.1148 | 0.1369 | 0.1555 | 0.1709 | 0.1833 | 0.1929 | 0.2004 | 0.2101 | 0.2151 | 0.2175 | 0.2186 | 0.2191 |
| | 2 | 9.01(-3) | 1.75(-2) | 2.51(-2) | 3.16(-2) | 3.67(-2) | 4.07(-2) | 4.35(-2) | 4.55(-2) | 4.69(-2) | 4.77(-2) | 4.85(-2) | 4.88(-2) | | | 4.88(-2) |
| | 3 | 2.82(-3) | 5.44(-3) | 7.68(-3) | 9.47(-3) | 1.08(-2) | 1.17(-2) | 1.23(-2) | 1.26(-2) | 1.28(-2) | 1.30(-2) | | | | | 1.30(-2) |
| | 4 | 9.20(-4) | 1.76(-3) | 2.44(-3) | 2.96(-3) | 3.31(-3) | 3.53(-3) | 3.66(-3) | 3.72(-3) | 3.76(-3) | 3.77(-3) | | | | | 3.77(-3) |
| | 5 | 3.07(-4) | 5.80(-4) | 7.96(-4) | 9.49(-4) | 1.05(-3) | 1.10(-3) | 1.13(-3) | 1.14(-3) | 1.15(-3) | | | | | | 1.15(-3) |
| | 6 | 1.04(-4) | 1.95(-4) | 2.64(-4) | 3.10(-4) | 3.36(-4) | 3.50(-4) | 3.56(-4) | 3.59(-4) | 3.60(-4) | | | | | | 3.60(-4) |
| | 7 | 3.56(-5) | 6.61(-5) | 8.84(-5) | 1.02(-4) | 1.10(-4) | 1.13(-4) | 1.15(-4) | 1.15(-4) | 1.15(-4) | | | | | | 1.15(-4) |
| | 8 | 1.23(-5) | 2.26(-5) | 2.99(-5) | 3.42(-5) | 3.63(-5) | 3.72(-5) | 3.75(-5) | 3.76(-5) | 3.77(-5) | | | | | | 3.77(-5) |
| | 9 | 4.28(-6) | 7.79(-6) | 1.02(-5) | 1.15(-5) | 1.21(-5) | 1.23(-5) | 1.24(-5) | 1.24(-5) | | | | | | | 1.24(-5) |
| | 10 | 1.49(-6) | 2.70(-6) | 3.48(-6) | 3.90(-6) | 4.07(-6) | 4.13(-6) | 4.15(-6) | 4.16(-6) | | | | | | | 4.16(-6) |

Values of $M(u,\beta)$ are equal to $W(u)$ for u greater than 10 and all values of $\beta$

The numbers in parenthesis are powers of 10 by which the other numbers are raised e.g., 8.74(-2) = 0.087

TABLE 1.—*CONTINUED*

| u | β | 2.0 | 2.2 | 2.4 | 2.6 | 2.8 | 3.0 | 3.2 | 3.4 | 3.6 | 3.8 | 4.0 | 4.2 | 4.4 | 4.6 | 4.8 | 5.0 | 5.2 | 5.4 | 5.6 | 5.8 | 6.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 2.8872 | 3.0593 | 3.2188 | 3.3675 | 3.5064 | 3.6369 | 3.7597 | 3.8757 | 3.9856 | 4.0900 | 4.1894 | 4.2842 | 4.3748 | 4.4616 | 4.5448 | 4.6248 | 4.7018 | 4.7760 | 4.8475 | 4.9167 | 4.9836 |
| $10^{-6}$ | 1 | 2.8827 | 3.0543 | 3.2134 | 3.3616 | 3.5001 | 3.6301 | 3.7525 | 3.8681 | 3.9775 | 4.0815 | 4.1804 | 4.2747 | 4.3649 | 4.4512 | 4.5340 | 4.6126 | 4.6901 | 4.7638 | 4.8349 | 4.9036 | 4.9700 |
| | 2 | 2.8809 | 3.0523 | 3.2112 | 3.3592 | 3.4975 | 3.6273 | 3.7495 | 3.8649 | 3.9742 | 4.0779 | 4.1766 | 4.2708 | 4.3608 | 4.4469 | 4.5295 | 4.6089 | 4.6852 | 4.7588 | 4.8297 | 4.8983 | 4.9644 |
| | 3 | 2.8794 | 3.0507 | 3.2095 | 3.3573 | 3.4955 | 3.6251 | 3.7472 | 3.8625 | 3.9716 | 4.0752 | 4.1738 | 4.2678 | 4.3576 | 4.4435 | 4.5261 | 4.6053 | 4.6815 | 4.7549 | 4.8257 | 4.8940 | 4.9601 |
| | 4 | 2.8782 | 3.0494 | 3.2080 | 3.3557 | 3.4938 | 3.6233 | 3.7453 | 3.8604 | 3.9694 | 4.0729 | 4.1714 | 4.2653 | 4.3550 | 4.4408 | 4.5232 | 4.6023 | 4.6784 | 4.7515 | 4.8223 | 4.8905 | 4.9566 |
| | 5 | 2.8772 | 3.0482 | 3.2067 | 3.3544 | 3.4923 | 3.6217 | 3.7436 | 3.8586 | 3.9675 | 4.0709 | 4.1692 | 4.2630 | 4.3526 | 4.4384 | 4.5206 | 4.5996 | 4.6756 | 4.7488 | 4.8193 | 4.8874 | 4.9533 |
| | 6 | 2.8762 | 3.0471 | 3.2056 | 3.3531 | 3.4910 | 3.6203 | 3.7420 | 3.8569 | 3.9658 | 4.0690 | 4.1673 | 4.2610 | 4.3505 | 4.4362 | 4.5183 | 4.5972 | 4.6731 | 4.7462 | 4.8166 | 4.8846 | 4.9504 |
| | 7 | 2.8753 | 3.0462 | 3.2045 | 3.3519 | 3.4897 | 3.6190 | 3.7406 | 3.8554 | 3.9642 | 4.0674 | 4.1655 | 4.2591 | 4.3486 | 4.4341 | 4.5162 | 4.5950 | 4.6708 | 4.7438 | 4.8141 | 4.8821 | 4.9477 |
| | 8 | 2.8745 | 3.0453 | 3.2035 | 3.3509 | 3.4886 | 3.6177 | 3.7393 | 3.8540 | 3.9627 | 4.0658 | 4.1639 | 4.2574 | 4.3467 | 4.4323 | 4.5142 | 4.5929 | 4.6686 | 4.7415 | 4.8118 | 4.8797 | 4.9452 |
| | 9 | 2.8737 | 3.0444 | 3.2026 | 3.3499 | 3.4876 | 3.6166 | 3.7380 | 3.8527 | 3.9613 | 4.0643 | 4.1623 | 4.2558 | 4.3450 | 4.4305 | 4.5124 | 4.5910 | 4.6665 | 4.7396 | 4.8097 | 4.8776 | 4.9429 |
| $10^{-5}$ | 1 | 2.8730 | 3.0436 | 3.2017 | 3.3489 | 3.4865 | 3.6156 | 3.7369 | 3.8515 | 3.9600 | 4.0629 | 4.1609 | 4.2542 | 4.3434 | 4.4288 | 4.5106 | 4.5892 | 4.6647 | 4.7375 | 4.8076 | 4.8753 | 4.9407 |
| | 2 | 2.8671 | 3.0371 | 3.1946 | 3.3412 | 3.4782 | 3.6066 | 3.7274 | 3.8414 | 3.9493 | 4.0517 | 4.1490 | 4.2418 | 4.3304 | 4.4152 | 4.4964 | 4.5744 | 4.6494 | 4.7215 | 4.7911 | 4.8582 | 4.9230 |
| | 3 | 2.8625 | 3.0321 | 3.1892 | 3.3353 | 3.4718 | 3.5998 | 3.7202 | 3.8337 | 3.9412 | 4.0431 | 4.1400 | 4.2323 | 4.3204 | 4.4048 | 4.4855 | 4.5631 | 4.6376 | 4.7093 | 4.7784 | 4.8450 | 4.9094 |
| | 4 | 2.8587 | 3.0279 | 3.1846 | 3.3304 | 3.4665 | 3.5941 | 3.7140 | 3.8272 | 3.9343 | 4.0358 | 4.1323 | 4.2243 | 4.3120 | 4.3960 | 4.4764 | 4.5535 | 4.6276 | 4.6989 | 4.7677 | 4.8339 | 4.8979 |
| | 5 | 2.8553 | 3.0242 | 3.1806 | 3.3260 | 3.4618 | 3.5890 | 3.7087 | 3.8215 | 3.9282 | 4.0294 | 4.1256 | 4.2172 | 4.3046 | 4.3882 | 4.4683 | 4.5451 | 4.6189 | 4.6899 | 4.7583 | 4.8242 | 4.8878 |
| | 6 | 2.8523 | 3.0209 | 3.1769 | 3.3220 | 3.4575 | 3.5844 | 3.7038 | 3.8163 | 3.9227 | 4.0236 | 4.1195 | 4.2108 | 4.2979 | 4.3812 | 4.4610 | 4.5375 | 4.6110 | 4.6816 | 4.7497 | 4.8153 | 4.8787 |
| | 7 | 2.8495 | 3.0178 | 3.1735 | 3.3184 | 3.4536 | 3.5802 | 3.6993 | 3.8116 | 3.9177 | 4.0183 | 4.1139 | 4.2049 | 4.2918 | 4.3748 | 4.4543 | 4.5305 | 4.6037 | 4.6741 | 4.7419 | 4.8072 | 4.8703 |
| | 8 | 2.8469 | 3.0149 | 3.1704 | 3.3150 | 3.4499 | 3.5763 | 3.6951 | 3.8071 | 3.9130 | 4.0133 | 4.1087 | 4.1994 | 4.2860 | 4.3688 | 4.4480 | 4.5240 | 4.5969 | 4.6670 | 4.7346 | 4.7997 | 4.8625 |
| | 9 | 2.8444 | 3.0123 | 3.1675 | 3.3118 | 3.4465 | 3.5727 | 3.6912 | 3.8030 | 3.9086 | 4.0087 | 4.1038 | 4.1943 | 4.2806 | 4.3632 | 4.4421 | 4.5178 | 4.5905 | 4.6604 | 4.7277 | 4.7926 | 4.8551 |
| $10^{-4}$ | 1 | 2.8421 | 3.0097 | 3.1647 | 3.3088 | 3.4433 | 3.5692 | 3.6875 | 3.7990 | 3.9044 | 4.0043 | 4.0992 | 4.1894 | 4.2756 | 4.3578 | 4.4366 | 4.5121 | 4.5845 | 4.6542 | 4.7212 | 4.7859 | 4.8482 |
| | 2 | 2.8234 | 2.9891 | 3.1423 | 3.2845 | 3.4171 | 3.5412 | 3.6576 | 3.7673 | 3.8708 | 3.9688 | 4.0618 | 4.1502 | 4.2345 | 4.3149 | 4.3918 | 4.4654 | 4.5360 | 4.6038 | 4.6690 | 4.7317 | 4.7922 |
| | 3 | 2.8091 | 2.9733 | 3.1251 | 3.2659 | 3.3970 | 3.5197 | 3.6347 | 3.7429 | 3.8450 | 3.9416 | 4.0332 | 4.1202 | 4.2030 | 4.2820 | 4.3574 | 4.4296 | 4.4988 | 4.5652 | 4.6289 | 4.6903 | 4.7493 |
| | 4 | 2.7970 | 2.9600 | 3.1106 | 3.2502 | 3.3801 | 3.5015 | 3.6154 | 3.7224 | 3.8233 | 3.9187 | 4.0090 | 4.0948 | 4.1764 | 4.2542 | 4.3285 | 4.3995 | 4.4674 | 4.5326 | 4.5952 | 4.6553 | 4.7133 |
| | 5 | 2.7864 | 2.9483 | 3.0978 | 3.2363 | 3.3652 | 3.4856 | 3.5984 | 3.7043 | 3.8042 | 3.8985 | 3.9878 | 4.0725 | 4.1531 | 4.2298 | 4.3030 | 4.3729 | 4.4399 | 4.5040 | 4.5655 | 4.6246 | 4.6814 |
| | 6 | 2.7768 | 2.9378 | 3.0863 | 3.2238 | 3.3518 | 3.4712 | 3.5830 | 3.6880 | 3.7869 | 3.8802 | 3.9686 | 4.0524 | 4.1320 | 4.2077 | 4.2800 | 4.3490 | 4.4150 | 4.4781 | 4.5387 | 4.5969 | 4.6527 |
| | 7 | 2.7679 | 2.9280 | 3.0757 | 3.2123 | 3.3394 | 3.4579 | 3.5688 | 3.6730 | 3.7710 | 3.8635 | 3.9509 | 4.0338 | 4.1126 | 4.1875 | 4.2588 | 4.3270 | 4.3921 | 4.4544 | 4.5141 | 4.5714 | 4.6264 |
| | 8 | 2.7597 | 2.9190 | 3.0658 | 3.2017 | 3.3279 | 3.4456 | 3.5557 | 3.6590 | 3.7562 | 3.8479 | 3.9345 | 4.0166 | 4.0945 | 4.1686 | 4.2392 | 4.3065 | 4.3708 | 4.4323 | 4.4912 | 4.5477 | 4.6019 |
| | 9 | 2.7519 | 2.9105 | 3.0565 | 3.1916 | 3.3171 | 3.4340 | 3.5434 | 3.6459 | 3.7423 | 3.8332 | 3.9191 | 4.0004 | 4.0776 | 4.1509 | 4.2207 | 4.2873 | 4.3508 | 4.4116 | 4.4697 | 4.5255 | 4.5789 |
| $10^{-3}$ | 1 | 2.7446 | 2.9024 | 3.0478 | 3.1821 | 3.3069 | 3.4231 | 3.5317 | 3.6335 | 3.7292 | 3.8194 | 3.9046 | 3.9852 | 4.0616 | 4.1342 | 4.2033 | 4.2691 | 4.3320 | 4.3920 | 4.4494 | 4.5045 | 4.5573 |
| | 2 | 2.6857 | 2.8377 | 2.9771 | 3.1056 | 3.2245 | 3.3349 | 3.4377 | 3.5337 | 3.6236 | 3.7080 | 3.7874 | 3.8623 | 3.9329 | 3.9998 | 4.0632 | 4.1233 | 4.1805 | 4.2349 | 4.2867 | 4.3360 | 4.3833 |
| | 3 | 2.6406 | 2.7881 | 2.9231 | 3.0471 | 3.1616 | 3.2675 | 3.3659 | 3.4575 | 3.5430 | 3.6230 | 3.6981 | 3.7686 | 3.8350 | 3.8975 | 3.9566 | 4.0125 | 4.0654 | 4.1156 | 4.1633 | 4.2084 | 4.2514 |
| | 4 | 2.6027 | 2.7464 | 2.8776 | 2.9980 | 3.1087 | 3.2110 | 3.3056 | 3.3936 | 3.4754 | 3.5518 | 3.6233 | 3.6902 | 3.7530 | 3.8120 | 3.8676 | 3.9199 | 3.9694 | 4.0161 | 4.0602 | 4.1020 | 4.1416 |
| | 5 | 2.5693 | 2.7098 | 2.8377 | 2.9548 | 3.0623 | 3.1613 | 3.2528 | 3.3376 | 3.4162 | 3.4894 | 3.5577 | 3.6215 | 3.6813 | 3.7372 | 3.7898 | 3.8391 | 3.8856 | 3.9293 | 3.9705 | 4.0094 | 4.0462 |
| | 6 | 2.5393 | 2.6767 | 2.8018 | 2.9159 | 3.0205 | 3.1166 | 3.2052 | 3.2871 | 3.3629 | 3.4334 | 3.4989 | 3.5599 | 3.6169 | 3.6702 | 3.7200 | 3.7667 | 3.8105 | 3.8517 | 3.8903 | 3.9267 | 3.9609 |
| | 7 | 2.5117 | 2.6464 | 2.7688 | 2.8802 | 2.9822 | 3.0757 | 3.1616 | 3.2409 | 3.3142 | 3.3821 | 3.4451 | 3.5036 | 3.5581 | 3.6090 | 3.6564 | 3.7007 | 3.7422 | 3.7810 | 3.8174 | 3.8515 | 3.8835 |
| | 8 | 2.4861 | 2.6183 | 2.7382 | 2.8472 | 2.9468 | 3.0377 | 3.1213 | 3.1982 | 3.2691 | 3.3346 | 3.3953 | 3.4516 | 3.5038 | 3.5524 | 3.5977 | 3.6398 | 3.6792 | 3.7159 | 3.7502 | 3.7823 | 3.8123 |
| | 9 | 2.4620 | 2.5920 | 2.7095 | 2.8162 | 2.9134 | 3.0022 | 3.0835 | 3.1582 | 3.2270 | 3.2903 | 3.3489 | 3.4030 | 3.4532 | 3.4998 | 3.5430 | 3.5832 | 3.6206 | 3.6554 | 3.6878 | 3.7181 | 3.7463 |
| $10^{-2}$ | 1 | 2.4394 | 2.5671 | 2.6825 | 2.7870 | 2.8820 | 2.9687 | 3.0480 | 3.1206 | 3.1873 | 3.2467 | 3.3052 | 3.3574 | 3.4057 | 3.4503 | 3.4917 | 3.5300 | 3.5656 | 3.5987 | 3.6294 | 3.6580 | 3.6845 |
| | 2 | 2.2587 | 2.3692 | 2.4676 | 2.5552 | 2.6337 | 2.7041 | 2.7673 | 2.8243 | 2.8756 | 2.9218 | 2.9637 | 3.0015 | 3.0357 | 3.0668 | 3.0946 | 3.1200 | 3.1430 | 3.1638 | 3.1827 | 3.1998 | 3.2153 |
| | 3 | 2.1233 | 2.2212 | 2.3072 | 2.3830 | 2.4498 | 2.5088 | 2.5610 | 2.6073 | 2.6482 | 2.6846 | 2.7168 | 2.7453 | 2.7707 | 2.7932 | 2.8131 | 2.8307 | 2.8464 | 2.8602 | 2.8724 | 2.8832 | 2.8928 |
| | 4 | 2.0117 | 2.0996 | 2.1759 | 2.2423 | 2.3000 | 2.3503 | 2.3942 | 2.4324 | 2.4658 | 2.4949 | 2.5202 | 2.5423 | 2.5615 | 2.5782 | 2.5927 | 2.6052 | 2.6161 | 2.6256 | 2.6337 | 2.6408 | 2.6468 |
| | 5 | 1.9166 | 1.9951 | 2.0634 | 2.1221 | 2.1725 | 2.2158 | 2.2531 | 2.2851 | 2.3125 | 2.3361 | 2.3563 | 2.3735 | 2.3883 | 2.4009 | 2.4116 | 2.4207 | 2.4284 | 2.4350 | 2.4405 | 2.4452 | 2.4491 |
| | 6 | 1.8307 | 1.9031 | 1.9645 | 2.0167 | 2.0610 | 2.0986 | 2.1305 | 2.1574 | 2.1802 | 2.1995 | 2.2157 | 2.2294 | 2.2408 | 2.2504 | 2.2584 | 2.2651 | 2.2706 | 2.2752 | 2.2790 | 2.2821 | 2.2846 |
| | 7 | 1.7543 | 1.8206 | 1.8760 | 1.9227 | 1.9618 | 1.9946 | 2.0220 | 2.0449 | 2.0640 | 2.0798 | 2.0929 | 2.1038 | 2.1127 | 2.1201 | 2.1261 | 2.1310 | 2.1350 | 2.1382 | 2.1408 | 2.1429 | 2.1446 |
| | 8 | 1.6848 | 1.7459 | 1.7959 | 1.8378 | 1.8725 | 1.9012 | 1.9249 | 1.9444 | 1.9604 | 1.9734 | 1.9841 | 1.9928 | 1.9998 | 2.0055 | 2.0101 | 2.0137 | 2.0166 | 2.0189 | 2.0207 | 2.0221 | 2.0233 |
| | 9 | 1.6210 | 1.6767 | 1.7226 | 1.7603 | 1.7912 | 1.8164 | 1.8370 | 1.8536 | 1.8671 | 1.8780 | 1.8867 | 1.8937 | 1.8992 | 1.9036 | 1.9071 | 1.9098 | 1.9119 | 1.9135 | 1.9148 | 1.9158 | 1.9165 |
| $10^{-1}$ | 1 | 1.5819 | 1.6133 | 1.6552 | 1.6892 | 1.7167 | 1.7389 | 1.7568 | 1.7711 | 1.7825 | 1.7915 | 1.7987 | 1.8043 | 1.8087 | 1.8121 | 1.8147 | 1.8168 | 1.8183 | 1.8195 | 1.8204 | 1.8211 | 1.8218 |
| | 2 | 1.1352 | 1.1596 | 1.1777 | 1.1909 | 1.2004 | 1.2073 | 1.2122 | 1.2166 | 1.2179 | 1.2195 | 1.2206 | 1.2213 | 1.2218 | 1.2221 | 1.2223 | 1.2224 | 1.2225 | 1.2226 | 1.2226 | 1.2226 | 1.2226 |
| | 3 | 0.8699 | 0.8823 | 0.8907 | 0.8962 | 0.8998 | 0.9021 | 0.9035 | 0.9044 | 0.9049 | 0.9053 | 0.9054 | 0.9056 | 0.9056 | 0.9056 | 0.9057 | | | | | | 0.9057 |
| | 4 | 0.6863 | 0.6928 | 0.6968 | 0.6992 | 0.7008 | 0.7014 | 0.7019 | 0.7021 | 0.7023 | 0.7023 | 0.7024 | | | | | | | | | | 0.7024 |
| | 5 | 0.5521 | 0.5556 | 0.5576 | 0.5587 | 0.5592 | 0.5595 | 0.5597 | 0.5597 | 0.5597 | | | | | | | | | | | | 0.5597 |
| | 6 | 0.4506 | 0.4525 | 0.4535 | 0.4540 | 0.4542 | 0.4543 | 0.4543 | | | | | | | | | | | | | | 0.4543 |
| | 7 | 0.3719 | 0.3729 | 0.3734 | 0.3735 | 0.3737 | 0.3737 | | | | | | | | | | | | | | | 0.3737 |
| | 8 | 0.3098 | 0.3102 | 0.3104 | 0.3105 | | | | | | | | | | | | | | | | | 0.3106 |
| | 9 | 0.2597 | 0.2600 | 0.2601 | 0.2602 | | | | | | | | | | | | | | | | | 0.2602 |
| | 1 | 0.2191 | 0.2193 | 0.2194 | | | | | | | | | | | | | | | | | | 0.2194 |

Values of M(u,β) are equal to W(u) for u greater than 1 and all values of β

| u | β | 6.0 | 6.2 | 6.4 | 6.6 | 6.8 | 7.0 | 7.2 | 7.4 | 7.6 | 7.8 | 8.0 | 8.2 | 8.4 | 8.6 | 8.8 | 9.0 | 9.2 | 9.4 | 9.6 | 9.8 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 4.9836 | 5.0482 | 5.1109 | 5.1718 | 5.2308 | 5.2882 | 5.3440 | 5.3983 | 5.4511 | 5.5026 | 5.5529 | 5.6019 | 5.6497 | 5.6965 | 5.7421 | 5.7868 | 5.8305 | 5.8733 | 5.9151 | 5.9562 | 5.9964 |
| $10^{-6}$ | 1 | 4.9700 | 5.0343 | 5.0965 | 5.1569 | 5.2155 | 5.2724 | 5.3278 | 5.3816 | 5.4340 | 5.4851 | 5.5349 | 5.5834 | 5.6308 | 5.6771 | 5.7223 | 5.7666 | 5.8098 | 5.8521 | 5.8935 | 5.9341 | 5.9739 |
| | 2 | 4.9644 | 5.0285 | 5.0905 | 5.1507 | 5.2091 | 5.2659 | 5.3210 | 5.3747 | 5.4269 | 5.4778 | 5.5274 | 5.5758 | 5.6230 | 5.6691 | 5.7141 | 5.7581 | 5.8012 | 5.8433 | 5.8846 | 5.9250 | 5.9645 |
| | 3 | 4.9601 | 5.0240 | 5.0860 | 5.1460 | 5.2043 | 5.2609 | 5.3159 | 5.3694 | 5.4215 | 5.4722 | 5.5217 | 5.5699 | 5.6170 | 5.6629 | 5.7078 | 5.7517 | 5.7946 | 5.8366 | 5.8777 | 5.9179 | 5.9573 |
| | 4 | 4.9566 | 5.0203 | 5.0821 | 5.1420 | 5.2002 | 5.2566 | 5.3115 | 5.3649 | 5.4169 | 5.4675 | 5.5168 | 5.5649 | 5.6119 | 5.6577 | 5.7025 | 5.7463 | 5.7890 | 5.8309 | 5.8719 | 5.9120 | 5.9513 |
| | 5 | 4.9533 | 5.0170 | 5.0787 | 5.1385 | 5.1965 | 5.2529 | 5.3077 | 5.3610 | 5.4128 | 5.4633 | 5.5126 | 5.5606 | 5.6074 | 5.6531 | 5.6978 | 5.7415 | 5.7841 | 5.8259 | 5.8668 | 5.9068 | 5.9460 |
| | 6 | 4.9504 | 5.0140 | 5.0756 | 5.1353 | 5.1933 | 5.2495 | 5.3042 | 5.3574 | 5.4092 | 5.4596 | 5.5087 | 5.5566 | 5.6034 | 5.6490 | 5.6936 | 5.7371 | 5.7797 | 5.8214 | 5.8621 | 5.9021 | 5.9412 |
| | 7 | 4.9477 | 5.0112 | 5.0728 | 5.1324 | 5.1902 | 5.2464 | 5.3010 | 5.3541 | 5.4058 | 5.4561 | 5.5052 | 5.5530 | 5.5996 | 5.6452 | 5.6897 | 5.7331 | 5.7756 | 5.8172 | 5.8579 | 5.8977 | 5.9367 |
| | 8 | 4.9452 | 5.0087 | 5.0701 | 5.1297 | 5.1874 | 5.2435 | 5.2981 | 5.3511 | 5.4027 | 5.4529 | 5.5019 | 5.5496 | 5.5962 | 5.6416 | 5.6860 | 5.7294 | 5.7718 | 5.8133 | 5.8539 | 5.8937 | 5.9326 |
| | 9 | 4.9429 | 5.0063 | 5.0676 | 5.1271 | 5.1848 | 5.2408 | 5.2953 | 5.3482 | 5.3997 | 5.4499 | 5.4988 | 5.5464 | 5.5929 | 5.6383 | 5.6826 | 5.7259 | 5.7683 | 5.8097 | 5.8502 | 5.8899 | 5.9287 |
| $10^{-5}$ | 1 | 4.9407 | 5.0040 | 5.0653 | 5.1247 | 5.1823 | 5.2383 | 5.2926 | 5.3455 | 5.3969 | 5.4470 | 5.4958 | 5.5434 | 5.5898 | 5.6352 | 5.6794 | 5.7226 | 5.7649 | 5.8063 | 5.8467 | 5.8863 | 5.9251 |
| | 2 | 4.9230 | 4.9857 | 5.0464 | 5.1052 | 5.1622 | 5.2176 | 5.2714 | 5.3238 | 5.3745 | 5.4240 | 5.4722 | 5.5192 | 5.5650 | 5.6097 | 5.6534 | 5.6961 | 5.7377 | 5.7785 | 5.8183 | 5.8573 | 5.8955 |
| | 3 | 4.9094 | 4.9716 | 5.0319 | 5.0902 | 5.1468 | 5.2017 | 5.2550 | 5.3069 | 5.3573 | 5.4063 | 5.4541 | 5.5006 | 5.5460 | 5.5903 | 5.6335 | 5.6757 | 5.7169 | 5.7572 | 5.7966 | 5.8351 | 5.8729 |
| | 4 | 4.8979 | 4.9598 | 5.0196 | 5.0776 | 5.1338 | 5.1883 | 5.2413 | 5.2927 | 5.3427 | 5.3914 | 5.4388 | 5.4849 | 5.5299 | 5.5738 | 5.6167 | 5.6585 | 5.6993 | 5.7392 | 5.7782 | 5.8164 | 5.8538 |
| | 5 | 4.8878 | 4.9493 | 5.0089 | 5.0665 | 5.1224 | 5.1766 | 5.2292 | 5.2803 | 5.3299 | 5.3783 | 5.4253 | 5.4711 | 5.5158 | 5.5594 | 5.6018 | 5.6433 | 5.6838 | 5.7234 | 5.7621 | 5.7999 | 5.8369 |
| | 6 | 4.8787 | 4.9399 | 4.9991 | 5.0565 | 5.1120 | 5.1659 | 5.2182 | 5.2690 | 5.3184 | 5.3664 | 5.4131 | 5.4587 | 5.5030 | 5.5463 | 5.5885 | 5.6296 | 5.6698 | 5.7091 | 5.7475 | 5.7850 | 5.8217 |
| | 7 | 4.8703 | 4.9312 | 4.9902 | 5.0472 | 5.1025 | 5.1561 | 5.2081 | 5.2586 | 5.3077 | 5.3555 | 5.4020 | 5.4472 | 5.4913 | 5.5342 | 5.5762 | 5.6171 | 5.6570 | 5.6960 | 5.7341 | 5.7713 | 5.8078 |
| | 8 | 4.8625 | 4.9232 | 4.9818 | 5.0386 | 5.0937 | 5.1470 | 5.1988 | 5.2490 | 5.2979 | 5.3453 | 5.3915 | 5.4365 | 5.4803 | 5.5231 | 5.5647 | 5.6053 | 5.6450 | 5.6837 | 5.7216 | 5.7586 | 5.7948 |
| | 9 | 4.8551 | 4.9156 | 4.9740 | 5.0305 | 5.0853 | 5.1384 | 5.1900 | 5.2400 | 5.2886 | 5.3358 | 5.3818 | 5.4265 | 5.4701 | 5.5126 | 5.5540 | 5.5943 | 5.6338 | 5.6723 | 5.7099 | 5.7466 | 5.7826 |
| $10^{-4}$ | 1 | 4.8482 | 4.9084 | 4.9666 | 5.0229 | 5.0775 | 5.1303 | 5.1816 | 5.2314 | 5.2798 | 5.3268 | 5.3725 | 5.4170 | 5.4604 | 5.5026 | 5.5438 | 5.5840 | 5.6231 | 5.6614 | 5.6988 | 5.7353 | 5.7710 |
| | 2 | 4.7922 | 4.8506 | 4.9069 | 4.9614 | 5.0141 | 5.0651 | 5.1145 | 5.1624 | 5.2089 | 5.2541 | 5.2980 | 5.3406 | 5.3821 | 5.4225 | 5.4619 | 5.5002 | 5.5375 | 5.5739 | 5.6095 | 5.6441 | 5.6780 |
| | 3 | 4.7493 | 4.8062 | 4.8612 | 4.9142 | 4.9655 | 5.0151 | 5.0631 | 5.1096 | 5.1547 | 5.1985 | 5.2409 | 5.2822 | 5.3223 | 5.3612 | 5.3992 | 5.4361 | 5.4720 | 5.5070 | 5.5411 | 5.5744 | 5.6069 |
| | 4 | 4.7133 | 4.7689 | 4.8227 | 4.8745 | 4.9246 | 4.9730 | 5.0198 | 5.0652 | 5.1091 | 5.1516 | 5.1929 | 5.2330 | 5.2719 | 5.3097 | 5.3464 | 5.3822 | 5.4169 | 5.4508 | 5.4837 | 5.5158 | 5.5471 |
| | 5 | 4.6814 | 4.7361 | 4.7888 | 4.8396 | 4.8886 | 4.9360 | 4.9818 | 5.0261 | 5.0689 | 5.1105 | 5.1507 | 5.1898 | 5.2276 | 5.2644 | 5.3001 | 5.3348 | 5.3686 | 5.4014 | 5.4333 | 5.4644 | 5.4947 |
| | 6 | 4.6527 | 4.7065 | 4.7582 | 4.8081 | 4.8562 | 4.9026 | 4.9475 | 4.9908 | 5.0327 | 5.0733 | 5.1127 | 5.1508 | 5.1877 | 5.2236 | 5.2583 | 5.2921 | 5.3249 | 5.3568 | 5.3879 | 5.4180 | 5.4474 |
| | 7 | 4.6264 | 4.6793 | 4.7301 | 4.7791 | 4.8264 | 4.8719 | 4.9159 | 4.9584 | 4.9995 | 5.0393 | 5.0777 | 5.1150 | 5.1511 | 5.1861 | 5.2200 | 5.2530 | 5.2850 | 5.3160 | 5.3462 | 5.3755 | 5.4041 |
| | 8 | 4.6019 | 4.6540 | 4.7040 | 4.7522 | 4.7987 | 4.8435 | 4.8867 | 4.9284 | 4.9687 | 5.0076 | 5.0453 | 5.0818 | 5.1171 | 5.1513 | 5.1845 | 5.2166 | 5.2478 | 5.2781 | 5.3075 | 5.3361 | 5.3639 |
| | 9 | 4.5789 | 4.6302 | 4.6796 | 4.7270 | 4.7727 | 4.8168 | 4.8592 | 4.9002 | 4.9397 | 4.9780 | 5.0149 | 5.0506 | 5.0852 | 5.1187 | 5.1512 | 5.1826 | 5.2131 | 5.2427 | 5.2714 | 5.2992 | 5.3263 |
| $10^{-3}$ | 1 | 4.5572 | 4.6078 | 4.6565 | 4.7032 | 4.7482 | 4.7915 | 4.8333 | 4.8736 | 4.9124 | 4.9500 | 4.9862 | 5.0213 | 5.0552 | 5.0880 | 5.1197 | 5.1505 | 5.1803 | 5.2092 | 5.2372 | 5.2644 | 5.2908 |
| | 2 | 4.3832 | 4.4282 | 4.4713 | 4.5125 | 4.5519 | 4.5898 | 4.6260 | 4.6609 | 4.6943 | 4.7264 | 4.7573 | 4.7870 | 4.8155 | 4.8430 | 4.8695 | 4.8950 | 4.9196 | 4.9433 | 4.9662 | 4.9882 | 5.0095 |
| | 3 | 4.2514 | 4.2923 | 4.3312 | 4.3684 | 4.4038 | 4.4376 | 4.4699 | 4.5007 | 4.5302 | 4.5584 | 4.5854 | 4.6113 | 4.6360 | 4.6597 | 4.6824 | 4.7042 | 4.7251 | 4.7452 | 4.7644 | 4.7829 | 4.8006 |
| | 4 | 4.1416 | 4.1702 | 4.2148 | 4.2487 | 4.2808 | 4.3114 | 4.3405 | 4.3682 | 4.3945 | 4.4197 | 4.4436 | 4.4664 | 4.4881 | 4.5089 | 4.5287 | 4.5476 | 4.5656 | 4.5829 | 4.5993 | 4.6150 | 4.6301 |
| | 5 | 4.0462 | 4.0809 | 4.1137 | 4.1448 | 4.1742 | 4.2021 | 4.2285 | 4.2535 | 4.2773 | 4.2998 | 4.3212 | 4.3415 | 4.3608 | 4.3792 | 4.3966 | 4.4131 | 4.4288 | 4.4438 | 4.4580 | 4.4715 | 4.4844 |
| | 6 | 3.9609 | 3.9932 | 4.0236 | 4.0523 | 4.0793 | 4.1048 | 4.1290 | 4.1517 | 4.1733 | 4.1936 | 4.2129 | 4.2311 | 4.2483 | 4.2646 | 4.2800 | 4.2946 | 4.3084 | 4.3214 | 4.3338 | 4.3455 | 4.3566 |
| | 7 | 3.8835 | 3.9138 | 3.9418 | 3.9684 | 3.9934 | 4.0168 | 4.0390 | 4.0598 | 4.0794 | 4.0978 | 4.1152 | 4.1316 | 4.1470 | 4.1616 | 4.1753 | 4.1882 | 4.2004 | 4.2119 | 4.2227 | 4.2329 | 4.2425 |
| | 8 | 3.8123 | 3.8404 | 3.8668 | 3.8914 | 3.9146 | 3.9362 | 3.9566 | 3.9756 | 3.9935 | 4.0103 | 4.0261 | 4.0409 | 4.0548 | 4.0678 | 4.0801 | 4.0916 | 4.1024 | 4.1125 | 4.1220 | 4.1309 | 4.1393 |
| | 9 | 3.7463 | 3.7720 | 3.7972 | 3.8202 | 3.8416 | 3.8617 | 3.8805 | 3.8980 | 3.9144 | 3.9297 | 3.9440 | 3.9575 | 3.9700 | 3.9817 | 3.9927 | 4.0029 | 4.0125 | 4.0215 | 4.0299 | 4.0377 | 4.0450 |
| $10^{-2}$ | 1 | 3.6845 | 3.7063 | 3.7323 | 3.7537 | 3.7737 | 3.7923 | 3.8096 | 3.8258 | 3.8408 | 3.8548 | 3.8679 | 3.8801 | 3.8914 | 3.9020 | 3.9119 | 3.9210 | 3.9296 | 3.9375 | 3.9449 | 3.9516 | 3.9582 |
| | 2 | 3.2153 | 3.2293 | 3.2419 | 3.2534 | 3.2638 | 3.2731 | 3.2816 | 3.2893 | 3.2961 | 3.3023 | 3.3079 | 3.3130 | 3.3175 | 3.3215 | 3.3252 | 3.3285 | 3.3314 | 3.3340 | 3.3364 | 3.3385 | 3.3403 |
| | 3 | 2.8928 | 2.9012 | 2.9086 | 2.9151 | 2.9209 | 2.9259 | 2.9303 | 2.9342 | 2.9376 | 2.9405 | 2.9431 | 2.9453 | 2.9473 | 2.9489 | 2.9504 | 2.9517 | 2.9528 | 2.9537 | 2.9545 | 2.9552 | 2.9558 |
| | 4 | 2.6468 | 2.6520 | 2.6565 | 2.6603 | 2.6636 | 2.6664 | 2.6688 | 2.6708 | 2.6725 | 2.6740 | 2.6752 | 2.6762 | 2.6771 | 2.6778 | 2.6784 | 2.6789 | 2.6793 | 2.6797 | 2.6800 | 2.6802 | 2.6804 |
| | 5 | 2.4491 | 2.4523 | 2.4551 | 2.4574 | 2.4593 | 2.4609 | 2.4622 | 2.4632 | 2.4641 | 2.4648 | 2.4654 | 2.4659 | 2.4663 | 2.4666 | 2.4669 | 2.4671 | 2.4673 | 2.4674 | 2.4675 | 2.4676 | 2.4677 |
| | 6 | 2.2846 | 2.2867 | 2.2884 | 2.2898 | 2.2909 | 2.2918 | 2.2926 | 2.2931 | 2.2936 | 2.2940 | 2.2943 | 2.2945 | 2.2947 | 2.2948 | 2.2949 | 2.2950 | 2.2951 | 2.2951 | 2.2952 | 2.2952 | 2.2953 |
| | 7 | 2.1446 | 2.1460 | 2.1470 | 2.1479 | 2.1486 | 2.1491 | 2.1495 | 2.1498 | 2.1500 | 2.1502 | 2.1504 | 2.1505 | 2.1506 | 2.1506 | 2.1507 | 2.1507 | 2.1508 | | | | 2.1508 |
| | 8 | 2.0233 | 2.0241 | 2.0248 | 2.0253 | 2.0257 | 2.0260 | 2.0263 | 2.0264 | 2.0266 | 2.0267 | 2.0267 | 2.0268 | 2.0268 | 2.0269 | | | | | | | 2.0269 |
| | 9 | 1.9165 | 1.9171 | 1.9175 | 1.9178 | 1.9181 | 1.9183 | 1.9184 | 1.9185 | 1.9186 | 1.9186 | 1.9186 | | | | | | | | | | 1.9186 |
| $10^{-1}$ | 1 | 1.8218 | 1.8219 | 1.8222 | 1.8224 | 1.8226 | 1.8227 | 1.8227 | 1.8228 | 1.8228 | 1.8229 | | | | | | | | | | | 1.8229 |
| | 2 | 1.2226 | 1.2226 | 1.2226 | | | | | | | | | | | | | | | | | | 1.2226 |

Values of M(u,β) are equal to W(u) for u greater than $2 \cdot 10^{-1}$ and all values of β

## TABLE 1.—*CONTINUED*

| $u$ | $\beta$ | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 | 42 | 44 | 46 | 48 | 50 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
|  | 0 | 5.9964 | 6.3595 | 6.6668 | 6.9333 | 7.1684 | 7.3789 | 7.5692 | 7.7431 | 7.9030 | 8.0511 | 8.1890 | 8.3180 | 8.4392 | 8.5535 | 8.6615 | 8.7641 | 8.8616 | 8.9546 | 9.0435 | 9.1286 | 9.2102 |
| $10^{-6}$ | 1 | 5.9739 | 6.3325 | 6.6353 | 6.8973 | 7.1279 | 7.3339 | 7.5197 | 7.6891 | 7.8445 | 7.9881 | 8.1215 | 8.2460 | 8.3627 | 8.4725 | 8.5761 | 8.6741 | 8.7671 | 8.8556 | 8.9400 | 9.0206 | 9.0977 |
|  | 2 | 5.9645 | 6.3213 | 6.6223 | 6.8823 | 7.1111 | 7.3152 | 7.4992 | 7.6667 | 7.8202 | 7.9620 | 8.0935 | 8.2161 | 8.3309 | 8.4388 | 8.5406 | 8.6367 | 8.7279 | 8.8145 | 8.8971 | 8.9758 | 9.0510 |
|  | 3 | 5.9573 | 6.3127 | 6.6122 | 6.8709 | 7.0982 | 7.3008 | 7.4834 | 7.6495 | 7.8016 | 7.9419 | 8.0720 | 8.1932 | 8.3066 | 8.4130 | 8.5133 | 8.6081 | 8.6978 | 8.7830 | 8.8641 | 8.9414 | 9.0152 |
|  | 4 | 5.9513 | 6.3054 | 6.6038 | 6.8612 | 7.0873 | 7.2887 | 7.4701 | 7.6350 | 7.7859 | 7.9250 | 8.0539 | 8.1739 | 8.2861 | 8.3913 | 8.4904 | 8.5839 | 8.6725 | 8.7565 | 8.8364 | 8.9125 | 8.9851 |
|  | 5 | 5.9460 | 6.2990 | 6.5963 | 6.8527 | 7.0777 | 7.2781 | 7.4584 | 7.6222 | 7.7721 | 7.9101 | 8.0379 | 8.1569 | 8.2680 | 8.3722 | 8.4702 | 8.5627 | 8.6502 | 8.7331 | 8.8119 | 8.8870 | 8.9586 |
|  | 6 | 5.9412 | 6.2932 | 6.5896 | 6.8450 | 7.0691 | 7.2685 | 7.4478 | 7.6106 | 7.7596 | 7.8966 | 8.0235 | 8.1415 | 8.2516 | 8.3549 | 8.4519 | 8.5435 | 8.6300 | 8.7120 | 8.7899 | 8.8640 | 8.9346 |
|  | 7 | 5.9367 | 6.2879 | 6.5834 | 6.8379 | 7.0611 | 7.2598 | 7.4381 | 7.6000 | 7.7481 | 7.8842 | 8.0102 | 8.1273 | 8.2366 | 8.3390 | 8.4352 | 8.5258 | 8.6115 | 8.6926 | 8.7696 | 8.8428 | 8.9126 |
|  | 8 | 5.9326 | 6.2830 | 6.5776 | 6.8313 | 7.0537 | 7.2514 | 7.4290 | 7.5901 | 7.7374 | 7.8727 | 7.9979 | 8.1142 | 8.2226 | 8.3242 | 8.4196 | 8.5094 | 8.5942 | 8.6745 | 8.7507 | 8.8231 | 8.8921 |
|  | 9 | 5.9287 | 6.2783 | 6.5722 | 6.8251 | 7.0467 | 7.2436 | 7.4205 | 7.5809 | 7.7273 | 7.8619 | 7.9863 | 8.1018 | 8.2095 | 8.3103 | 8.4049 | 8.4940 | 8.5781 | 8.6576 | 8.7330 | 8.8047 | 8.8729 |
| $10^{-5}$ | 1 | 5.9251 | 6.2739 | 6.5671 | 6.8193 | 7.0402 | 7.2363 | 7.4125 | 7.5721 | 7.7178 | 7.8517 | 7.9753 | 8.0901 | 8.1971 | 8.2972 | 8.3910 | 8.4794 | 8.5628 | 8.6416 | 8.7163 | 8.7872 | 8.8547 |
|  | 2 | 5.8955 | 6.2385 | 6.5257 | 6.7720 | 6.9870 | 7.1773 | 7.3476 | 7.5013 | 7.6412 | 7.7692 | 7.8871 | 7.9960 | 8.0972 | 8.1914 | 8.2795 | 8.3621 | 8.4397 | 8.5127 | 8.5817 | 8.6469 | 8.7087 |
|  | 3 | 5.8729 | 6.2113 | 6.4940 | 6.7358 | 6.9463 | 7.1321 | 7.2979 | 7.4472 | 7.5826 | 7.7061 | 7.8195 | 7.9240 | 8.0208 | 8.1106 | 8.1943 | 8.2725 | 8.3458 | 8.4145 | 8.4792 | 8.5401 | 8.5976 |
|  | 4 | 5.8538 | 6.1884 | 6.4673 | 6.7053 | 6.9120 | 7.0940 | 7.2561 | 7.4016 | 7.5332 | 7.6531 | 7.7627 | 7.8636 | 7.9566 | 8.0428 | 8.1229 | 8.1975 | 8.2671 | 8.3322 | 8.3933 | 8.4507 | 8.5047 |
|  | 5 | 5.8369 | 6.1682 | 6.4438 | 6.6785 | 6.8818 | 7.0606 | 7.2193 | 7.3615 | 7.4899 | 7.6065 | 7.7129 | 7.8105 | 7.9003 | 7.9833 | 8.0602 | 8.1316 | 8.1982 | 8.2602 | 8.3182 | 8.3725 | 8.4235 |
|  | 6 | 5.8217 | 6.1500 | 6.4225 | 6.6542 | 6.8546 | 7.0303 | 7.1861 | 7.3253 | 7.4508 | 7.5644 | 7.6679 | 7.7626 | 7.8496 | 7.9298 | 8.0038 | 8.0725 | 8.1362 | 8.1955 | 8.2508 | 8.3024 | 8.3507 |
|  | 7 | 5.8078 | 6.1332 | 6.4030 | 6.6319 | 6.8296 | 7.0025 | 7.1556 | 7.2921 | 7.4149 | 7.5259 | 7.6267 | 7.7188 | 7.8032 | 7.8807 | 7.9522 | 8.0183 | 8.0795 | 8.1364 | 8.1892 | 8.2384 | 8.2843 |
|  | 8 | 5.7948 | 6.1177 | 6.3848 | 6.6112 | 6.8063 | 6.9767 | 7.1272 | 7.2613 | 7.3815 | 7.4901 | 7.5885 | 7.6781 | 7.7601 | 7.8353 | 7.9044 | 7.9682 | 8.0271 | 8.0817 | 8.1323 | 8.1792 | 8.2229 |
|  | 9 | 5.7826 | 6.1030 | 6.3678 | 6.5917 | 6.7844 | 6.9525 | 7.1007 | 7.2324 | 7.3503 | 7.4565 | 7.5527 | 7.6401 | 7.7198 | 7.7928 | 7.8597 | 7.9214 | 7.9782 | 8.0306 | 8.0791 | 8.1241 | 8.1658 |
| $10^{-4}$ | 1 | 5.7710 | 6.0892 | 6.3317 | 6.5734 | 6.7638 | 6.9296 | 7.0756 | 7.2051 | 7.3208 | 7.4249 | 7.5189 | 7.6042 | 7.6818 | 7.7527 | 7.8177 | 7.8773 | 7.9321 | 7.9826 | 8.0292 | 8.0723 | 8.1122 |
|  | 2 | 5.6780 | 5.9778 | 6.2221 | 6.4257 | 6.5982 | 6.7463 | 6.8747 | 6.9869 | 7.0856 | 7.1729 | 7.2504 | 7.3194 | 7.3811 | 7.4364 | 7.4861 | 7.5307 | 7.5709 | 7.6072 | 7.6399 | 7.6695 | 7.6962 |
|  | 3 | 5.6069 | 5.8928 | 6.1233 | 6.3133 | 6.4725 | 6.6075 | 6.7231 | 6.8227 | 6.9091 | 6.9844 | 7.0502 | 7.1080 | 7.1588 | 7.2035 | 7.2429 | 8.2777 | 7.3084 | 7.3356 | 7.3597 | 7.3809 | 7.3997 |
|  | 4 | 5.5471 | 5.8214 | 6.0406 | 6.2194 | 6.3677 | 6.4920 | 6.5972 | 6.6868 | 6.7635 | 6.8294 | 6.8862 | 6.9353 | 6.9778 | 7.0146 | 7.0465 | 7.0742 | 7.0982 | 7.1191 | 7.1371 | 7.1528 | 7.1663 |
|  | 5 | 5.4947 | 5.7589 | 5.9681 | 6.1374 | 6.2763 | 6.3915 | 6.4880 | 6.5692 | 6.6378 | 6.6961 | 6.7456 | 6.7878 | 6.8237 | 6.8544 | 6.8805 | 6.9029 | 6.9219 | 6.9380 | 6.9518 | 6.9635 | 6.9734 |
|  | 6 | 5.4474 | 5.7026 | 5.9031 | 6.0638 | 6.1945 | 6.3019 | 6.3908 | 6.4648 | 6.5266 | 6.5784 | 6.6218 | 6.6583 | 6.6890 | 6.7147 | 6.7363 | 6.7545 | 6.7696 | 6.7823 | 6.7929 | 6.8017 | 6.8090 |
|  | 7 | 5.4041 | 5.6511 | 5.8437 | 5.9967 | 6.1201 | 6.2205 | 6.3027 | 6.3704 | 6.4263 | 6.4726 | 6.5109 | 6.5427 | 6.5690 | 6.5907 | 6.6087 | 6.6235 | 6.6357 | 6.6457 | 6.6539 | 6.6605 | 6.6661 |
|  | 8 | 5.3639 | 5.6034 | 5.7887 | 5.9348 | 6.0515 | 6.1456 | 6.2219 | 6.2841 | 6.3349 | 6.3763 | 6.4103 | 6.4380 | 6.4607 | 6.4791 | 6.4942 | 6.5063 | 6.5162 | 6.5241 | 6.5305 | 6.5357 | 6.5397 |
|  | 9 | 5.3263 | 5.5588 | 5.7374 | 5.8771 | 5.9878 | 6.0762 | 6.1472 | 6.2044 | 6.2506 | 6.2879 | 6.3181 | 6.3424 | 6.3620 | 6.3777 | 6.3903 | 6.4004 | 6.4084 | 6.4147 | 6.4197 | 6.4236 | 6.4267 |
| $10^{-3}$ | 1 | 5.2908 | 5.5168 | 5.6892 | 5.8230 | 5.9281 | 6.0113 | 6.0775 | 6.1303 | 6.1724 | 6.2061 | 6.2330 | 6.2543 | 6.2713 | 6.2848 | 6.2954 | 6.3037 | 6.3102 | 6.3153 | 6.3192 | 6.3222 | 6.3246 |
|  | 2 | 5.0095 | 5.1861 | 5.3123 | 5.4037 | 5.4701 | 5.5184 | 5.5534 | 5.5788 | 5.5970 | 5.6101 | 5.6193 | 5.6257 | 5.6302 | 5.6333 | 5.6354 | 5.6368 | 5.6377 | 5.6383 | 5.6387 | 5.6390 | 5.6391 |
|  | 3 | 4.8006 | 4.9437 | 5.0402 | 5.1056 | 5.1498 | 5.1795 | 5.1993 | 5.2124 | 5.2208 | 5.2263 | 5.2297 | 5.2318 | 5.2331 | 5.2339 | 5.2343 | 5.2346 | 5.2347 | 5.2348 | 5.2349 |  | 5.2349 |
|  | 4 | 4.6301 | 4.7481 | 4.8235 | 4.8714 | 4.9017 | 4.9205 | 4.9320 | 4.9390 | 4.9430 | 4.9454 | 4.9467 | 4.9474 | 4.9478 | 4.9480 | 4.9481 |  |  |  |  |  | 4.9481 |
|  | 5 | 4.4844 | 4.5830 | 4.6428 | 4.6783 | 4.6993 | 4.7114 | 4.7183 | 4.7220 | 4.7241 | 4.7251 | 4.7256 | 4.7259 | 4.7260 | 4.7260 |  |  |  |  |  |  | 4.7260 |
|  | 6 | 4.3566 | 4.4396 | 4.4872 | 4.5140 | 4.5288 | 4.5367 | 4.5409 | 4.5429 | 4.5439 | 4.5444 | 4.5446 | 4.5447 |  |  |  |  |  |  |  |  | 4.5447 |
|  | 7 | 4.2425 | 4.3128 | 4.3510 | 4.3714 | 4.3819 | 4.3871 | 4.3896 | 4.3908 | 4.3913 | 4.3915 | 4.3916 |  |  |  |  |  |  |  |  |  | 4.3916 |
|  | 8 | 4.1393 | 4.1991 | 4.2300 | 4.2455 | 4.2530 | 4.2565 | 4.2580 | 4.2587 | 4.2589 | 4.2590 | 4.2591 |  |  |  |  |  |  |  |  |  | 4.2591 |
|  | 9 | 4.0450 | 4.0961 | 4.1212 | 4.1331 | 4.1385 | 4.1408 | 4.1417 | 4.1421 | 4.1422 | 4.1423 |  |  |  |  |  |  |  |  |  |  | 4.1423 |
| $10^{-2}$ | 1 | 3.9582 | 4.0020 | 4.0224 | 4.0316 | 4.0355 | 4.0370 | 4.0376 | 4.0378 |  |  |  |  |  |  |  |  |  |  |  |  | 4.0378 |
|  | 2 | 3.3403 | 3.3507 | 3.3537 | 3.3545 | 3.3547 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 3.3547 |
|  | 3 | 2.9558 | 2.9585 | 2.9590 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 2.9590 |
|  | 4 | 2.6804 | 2.6812 | 2.6812 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 2.6812 |
|  | 5 | 2.4677 | 2.4679 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 2.4679 |
|  | 6 | 2.2952 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 2.2952 |

Values of $M(u,\beta)$ are equal to $W(u)$ for $u$ greater than $6 \cdot 10^{-2}$ and all values of $\beta$

| $u$ | $\beta$ | 50 | 52 | 54 | 56 | 58 | 60 | 62 | 64 | 66 | 68 | 70 | 72 | 74 | 76 | 78 | 80 | 82 | 84 | 86 | 88 | 90 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
|  | 0 | 9.2102 | 9.2886 | 9.3641 | 9.4368 | 9.5069 | 9.5747 | 9.6403 | 9.7037 | 9.7653 | 9.8249 | 9.8829 | 3.9392 | 9.9940 | 10.0473 | 10.0992 | 10.1498 | 10.1992 | 10.2474 | 10.2944 | 10.3404 | 10.3853 |
| $10^{-6}$ | 1 | 9.0977 | 9.1716 | 9.2426 | 9.3108 | 9.3765 | 9.4398 | 9.5008 | 9.5598 | 9.6168 | 9.6720 | 9.7255 | 9.7773 | 9.8276 | 9.8764 | 9.9239 | 9.9700 | 10.0148 | 10.0585 | 10.1011 | 10.1425 | 10.1830 |
|  | 2 | 9.0510 | 9.1231 | 9.1922 | 9.2585 | 9.3223 | 9.3838 | 9.4430 | 9.5001 | 9.5553 | 9.6086 | 9.6602 | 9.7102 | 9.7586 | 9.8056 | 9.8512 | 9.8955 | 9.9385 | 9.9803 | 10.0210 | 10.0606 | 10.0992 |
|  | 3 | 9.0152 | 9.0859 | 9.1535 | 9.2185 | 9.2809 | 9.3409 | 9.3986 | 9.4543 | 9.5081 | 9.5600 | 9.6102 | 9.6588 | 9.7058 | 9.7513 | 9.7955 | 9.8384 | 9.8800 | 9.9204 | 9.9597 | 9.9979 | 10.0351 |
|  | 4 | 8.9851 | 9.0545 | 9.1210 | 9.1847 | 9.2459 | 9.3047 | 9.3613 | 9.4158 | 9.4684 | 9.5191 | 9.5681 | 9.6155 | 9.6613 | 9.7057 | 9.7487 | 9.7904 | 9.8308 | 9.8700 | 9.9081 | 9.9452 | 9.9812 |
|  | 5 | 8.9586 | 9.0269 | 9.0924 | 9.1550 | 9.2152 | 9.2729 | 9.3285 | 9.3819 | 9.4335 | 9.4831 | 9.5311 | 9.5774 | 9.6222 | 9.6655 | 9.7075 | 9.7481 | 9.7875 | 9.8257 | 9.8628 | 9.8988 | 9.9338 |
|  | 6 | 8.9346 | 9.0020 | 9.0665 | 9.1282 | 9.1874 | 9.2442 | 9.2988 | 9.3513 | 9.4019 | 9.4507 | 9.4977 | 9.5431 | 9.5869 | 9.6293 | 9.6703 | 9.7101 | 9.7485 | 9.7858 | 9.8220 | 9.8571 | 9.8911 |
|  | 7 | 8.9126 | 8.9791 | 9.0427 | 9.1036 | 9.1619 | 9.2179 | 9.2716 | 9.3232 | 9.3730 | 9.4208 | 9.4670 | 9.5115 | 9.5545 | 9.5961 | 9.6362 | 9.6751 | 9.7127 | 9.7492 | 9.7845 | 9.8187 | 9.8519 |
|  | 8 | 8.8921 | 8.9578 | 9.0206 | 9.0807 | 9.1382 | 9.1933 | 9.2463 | 9.2971 | 9.3460 | 9.3931 | 9.4385 | 9.4822 | 9.5244 | 9.5652 | 9.6046 | 9.6426 | 9.6795 | 9.7151 | 9.7497 | 9.7831 | 9.8156 |
|  | 9 | 8.8729 | 8.9378 | 9.0000 | 9.0592 | 9.1160 | 9.1703 | 9.2225 | 9.2726 | 9.3208 | 9.3671 | 9.4118 | 9.4548 | 9.4962 | 9.5362 | 9.5749 | 9.6122 | 9.6483 | 9.6833 | 9.7171 | 9.7498 | 9.7815 |
| $10^{-5}$ | 1 | 8.8547 | 8.9190 | 8.9803 | 9.0389 | 9.0949 | 9.1486 | 9.2001 | 9.2495 | 9.2970 | 9.3426 | 9.3865 | 9.4288 | 9.4696 | 9.5089 | 9.5469 | 9.5835 | 9.6189 | 9.6532 | 9.6863 | 9.7183 | 9.7494 |
|  | 2 | 8.7087 | 8.7673 | 8.8229 | 8.8759 | 8.9263 | 8.9743 | 9.0202 | 9.0640 | 9.1059 | 9.1461 | 9.1845 | 9.2213 | 9.2566 | 9.2905 | 9.3230 | 9.3542 | 9.3843 | 9.4132 | 9.4410 | 9.4677 | 9.4935 |
|  | 3 | 8.5976 | 8.6519 | 8.7033 | 8.7521 | 8.7983 | 8.8422 | 8.8839 | 8.9237 | 8.9615 | 8.9975 | 9.0319 | 9.0647 | 9.0960 | 9.1260 | 9.1546 | 9.1819 | 9.2081 | 9.2332 | 9.2572 | 9.2802 | 9.3023 |
|  | 4 | 8.5047 | 8.5555 | 8.6035 | 8.6488 | 8.6916 | 8.7321 | 8.7705 | 8.8069 | 8.8414 | 8.8742 | 8.9053 | 8.9349 | 8.9630 | 8.9898 | 9.0153 | 9.0396 | 9.0628 | 9.0848 | 9.1059 | 9.1260 | 9.1451 |
|  | 5 | 8.4235 | 8.4713 | 8.5163 | 8.5587 | 8.5986 | 8.6362 | 8.6717 | 8.7053 | 8.7370 | 8.7671 | 8.7955 | 8.8224 | 8.8479 | 8.8721 | 8.8950 | 8.9167 | 8.9374 | 8.9570 | 8.9756 | 8.9932 | 9.0100 |
|  | 6 | 8.3507 | 8.3959 | 8.4383 | 8.4780 | 8.5154 | 8.5505 | 8.5836 | 8.6147 | 8.6440 | 8.6716 | 8.6977 | 8.7223 | 8.7455 | 8.7675 | 8.7882 | 8.8078 | 8.8263 | 8.8438 | 8.8603 | 8.8760 | 8.8908 |
|  | 7 | 8.2843 | 8.3271 | 8.3672 | 8.4046 | 8.4397 | 8.4726 | 8.5034 | 8.5324 | 8.5596 | 8.5851 | 8.6091 | 8.6317 | 8.6530 | 8.6730 | 8.6918 | 8.7095 | 8.7262 | 8.7419 | 8.7567 | 8.7706 | 8.7838 |
|  | 8 | 8.2229 | 8.2636 | 8.3016 | 8.3370 | 8.3700 | 8.4009 | 8.4297 | 8.4568 | 8.4821 | 8.5057 | 8.5279 | 8.5487 | 8.5682 | 8.5865 | 8.6036 | 8.6197 | 8.6348 | 8.6490 | 8.6623 | 8.6747 | 8.6864 |
|  | 9 | 8.1658 | 8.2045 | 8.2405 | 8.2740 | 8.3052 | 8.3343 | 8.3614 | 8.3866 | 8.4102 | 8.4322 | 8.4528 | 8.4720 | 8.4899 | 8.5067 | 8.5223 | 8.5370 | 8.5507 | 8.5635 | 8.5754 | 8.5866 | 8.5970 |
| $10^{-4}$ | 1 | 8.1122 | 8.1491 | 8.1833 | 8.2151 | 8.2446 | 8.2720 | 8.2974 | 8.3211 | 8.3431 | 8.3636 | 8.3827 | 8.4005 | 8.4170 | 8.4324 | 8.4468 | 8.4601 | 8.4726 | 8.4842 | 8.4949 | 8.5050 | 8.5143 |
|  | 2 | 7.6962 | 7.7203 | 7.7421 | 7.7618 | 7.7797 | 7.7958 | 7.8104 | 7.8236 | 7.8355 | 7.8463 | 7.8560 | 7.8648 | 7.8727 | 7.8798 | 7.8862 | 7.8920 | 7.8972 | 7.9019 | 7.9061 | 7.9098 | 7.9132 |
|  | 3 | 7.3997 | 7.4163 | 7.4309 | 7.4439 | 7.4553 | 7.4654 | 7.4742 | 7.4820 | 7.4889 | 7.4949 | 7.5002 | 7.5048 | 7.5088 | 7.5124 | 7.5154 | 7.5181 | 7.5204 | 7.5225 | 7.5242 | 7.5258 | 7.5271 |
|  | 4 | 7.1663 | 7.1780 | 7.1881 | 7.1968 | 7.2043 | 7.2108 | 7.2163 | 7.2211 | 7.2251 | 7.2286 | 7.2315 | 7.2341 | 7.2362 | 7.2380 | 7.2395 | 7.2408 | 7.2419 | 7.2428 | 7.2436 | 7.2442 | 7.2447 |
|  | 5 | 6.9734 | 6.9818 | 6.9888 | 6.9948 | 6.9998 | 7.0040 | 7.0075 | 7.0105 | 7.0129 | 7.0150 | 7.0167 | 7.0181 | 7.0192 | 7.0202 | 7.0209 | 7.0216 | 7.0221 | 7.0225 | 7.0228 | 7.0231 | 7.0233 |
|  | 6 | 6.8090 | 6.8151 | 6.8201 | 6.8242 | 6.8276 | 6.8304 | 6.8327 | 6.8345 | 6.8360 | 6.8372 | 6.8382 | 6.8390 | 6.8396 | 6.8401 | 6.8405 | 6.8408 | 6.8411 | 6.8413 | 6.8414 | 6.8416 | 6.8417 |
|  | 7 | 6.6661 | 6.6705 | 6.6741 | 6.6770 | 6.6793 | 6.6811 | 6.6826 | 6.6838 | 6.6847 | 6.6854 | 6.6860 | 6.6864 | 6.6868 | 6.6871 | 6.6873 | 6.6874 | 6.6875 | 6.6876 | 6.6877 | 6.6878 | 6.6878 |
|  | 8 | 6.5397 | 6.5430 | 6.5456 | 6.5476 | 6.5493 | 6.5504 | 6.5514 | 6.5521 | 6.5527 | 6.5531 | 6.5535 | 6.5537 | 6.5539 | 6.5541 | 6.5542 | 6.5543 | 6.5543 | 6.5544 | 6.5544 | 6.5544 | 6.5544 |
|  | 9 | 6.4267 | 6.4291 | 6.4310 | 6.4324 | 6.4335 | 6.4343 | 6.4350 | 6.4354 | 6.4358 | 6.4361 | 6.4363 | 6.4364 | 6.4365 | 6.4366 | 6.4366 | 6.4367 | 6.4367 | 6.4367 | 6.4368 |  | 6.4368 |
| $10^{-3}$ | 1 | 6.3246 | 6.3263 | 6.3277 | 6.3287 | 6.3294 | 6.3300 | 6.3304 | 6.3307 | 6.3310 | 6.3311 | 6.3312 | 6.3313 | 6.3314 | 6.3314 | 6.3315 |  |  |  |  |  | 6.3315 |
|  | 2 | 5.6391 | 5.6392 | 5.6393 | 5.6393 | 5.6393 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 5.6393 |

Values of $M(u,\beta)$ are equal to $W(u)$ for $u$ greater than $2 \cdot 10^{-3}$ and all values of $\beta$

infinitely deep, Eq. 2a reduces to

$$s = \frac{Q}{8\,\pi\,K\,(l - d)}\;E\left(u, \frac{l}{r}, \frac{d}{r}, \frac{z}{r}\right) \;\ldots\ldots\ldots\ldots\ldots \quad (7)$$

in which the function E is given by the four M terms of Eq. 2a.

Eq. 7 states, in effect, that in the initial period of pumping, the drawdown around a partially penetrating well would be the same as though the aquifer were infinitely deep. The length of the initial period depends on the penetration depth of the pumped well, the depth of the observation point, and the thickness of the formation, as well as the hydraulic properties of the squifer.

*Equation of Drawdown for Relatively Large Values of Time.*—It can be shown (see tables of $W(u,y)$) that for $u < ((\pi r/b)^2/20)$, the function $W(u, n\,\pi\,r/b)$ can, for all practical purposes, be replaced by $2\,K_o\,(n\,\pi\,r/b)$; in which case the series in Eq. 1 becomes independent of time. Hence, for $u < (1/2)(r/b)^2$, that is, $t\quad(b^2S_s/(2\,K))$, Eq. 1 becomes

$$s = \frac{Q}{4\,\pi\,K\,b}\left[\,W(u) + f_s\left(\frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}\right)\right] \;\ldots\ldots\ldots\ldots\ldots \quad (8a)$$

in which

$$f_s = \frac{4\,b}{\pi(l - d)} \sum_{n=1}^{\infty} \frac{1}{n}\,K_o\left(\frac{n\,\pi\,r}{b}\right)\left(\sin\frac{n\,\pi\,l}{b} - \sin\frac{n\,\pi d}{b}\right)\cos\frac{n\pi\,z}{b} \;\ldots\ldots \quad (8b)$$

in which $K_o$ is the zero-order modified Bessel function of the second kind.

Eq. 8a shows that in this range of time, the rate of change of drawdown is the same as though the pumped well completely penetrated the aquifer. In other words, the effect of partial penetration on the drawdown has attained its maximum value.

## AVERAGE DRAWDOWN IN OBSERVATION WELLS

The water level in an observation well reflects the average drawdown in the aquifer profile that is occupied by the screened portion (or perforated section of the casing) of the well. The average drawdown $\overline{s}$ in an observation well screened between the depths $l'$ and $d'$ $(l' > d')$ can be obtained by integrating the equation of drawdown in piezometers with respect to z between the limits $d'$ and $l'$, and then dividing the result by $(l' - d')$. If this operation is performed on Eq. 1, the result immediately can be seen to be:

$$\overline{s} = \frac{Q}{4\,\pi\,K b}\left[\,W(u), + \overline{f}\left(u, \frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{l'}{b}, \frac{d'}{b}\right)\right] \;\ldots\ldots\ldots \quad (9a)$$

in which

$$\overline{f} = \frac{2\,b^2}{\pi^2(l - d)(l' - d)} \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\sin\frac{n\,\pi\,l}{b} - \sin\frac{n\,\pi\,d}{b}\right)$$

$$\left(\sin\frac{n\,\pi\,l'}{b} - \sin\frac{n\,\pi\,d'}{b}\right) W\left(u, \frac{n\,\pi\,r}{b}\right) \;\ldots\ldots\ldots \quad (9b$$

The result of averaging Eq. 2a is rather complicated and will not be given here. Of interest in this result, however, is the part representing the average drawdown for relatively small values of time. This part is given below.

*Equation of Average Drawdown for Relatively Small Values of Time.*—For $t < \left[(2\,b - l - l')^2\,S_s\,/(20\,K)\right]$, the f' terms of Eq. 2a drop out for all practical purposes. The result of averaging the remaining four M terms is given by

$$\bar{s} = \frac{Q}{8\,\pi\,K(l - d)(l' - d')}\left[F\left(u, \frac{l + l'}{r}, \frac{l - l'}{r}\right) - F\left(u, \frac{d + l'}{r}, \frac{d - l'}{r}\right)\right.$$
$$\left. + F\left(u, \frac{d + d'}{r}, \frac{d - d'}{r}\right) - F\left(u, \frac{l + d'}{r}, \frac{l - d'}{r}\right)\right] \quad \ldots\ldots \quad (10a)$$

in which

$$F(u, \beta, \alpha) = r\left\{\beta\,M(u, \beta) - \alpha\,M(u, \alpha)\right.$$
$$\left. + 2\left[\sqrt{y}\,\mathrm{erfc}(\sqrt{y\,u}) - x\,\mathrm{erfc}\,(\sqrt{x\,u}) + \frac{e^{-xu} - e^{-yu}}{\sqrt{\pi\,u}}\right]\right\} \quad \ldots\ldots \quad (10b)$$

in which $x = 1 + \beta^2$ and $y = 1 + \alpha^2$, erfc is the complementary error function (1 - erf), and M is the function defined by Eq. 3a.

Eq. 10a also gives the average drawdown in observation wells tapping an infinitely deep aquifer for the whole range of pumping time.

For $d = d' = 0$, Eq. 7 as well as Eq. 10, will reduce to the corresponding special cases obtained by K. F. Saad[11] in his treatment of flow in thick artesian aquifers.

Computations in Eq. 10a show that if $(l'/l) < 2$, results sufficiently accurate for practical application can be obtained by using the following approximate equation

$$\bar{s} \approx \frac{Q}{8\,\pi\,K(l - d)}\,\bar{E}\left(u, \frac{l}{r}, \frac{d}{r}, \frac{l'}{r}, \frac{d'}{r}\right) \quad \ldots\ldots\ldots \quad (11)$$

in which $\bar{E}$ is the value of the function E of Eq. 7, in which the value of z is replaced by $(1/2)\,(l' + d')$. In other words, the average drawdown in an observation well screened between the depths l' and d' can be approximated by the average of drawdowns registered in two piezometers whose penetration depths are l' and d' respectively, provided that $(l'/l) < 2$.

Also, if $(r/l) > 1$ and $(l'/l) < 1$, the average drawdown in the observation well can, for all practical purposes, be taken as that given by Eq. 7, with the value of z arbitrarily chosen between l' and zero. The choice is generally made so as to simplify the equation, which in certain cases may take the following form:

$$\bar{s} \approx c\,M(u, \beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

in which case the equation is valid for $t < (2\,b - r\,\beta)^2 S_s/(20\,K)$, c and $\beta$ being constants that depend on the parameters of the flow system under consideration. For example, if $l = 3\,d$, a choice of $z = d$ will reduce Eq. 7 to the following

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[11] "Nonsteady Flow Toward Wells Which Partially Penetrate Thick Artesian Aquifers," by K. F. Saad, thesis presented to the New Mexico Institute of Mining and Technology, at Socorro, N. M., in 1960, in partial fulfilment of the requirements of the degree of Master of Science.

$$\overline{s} \approx \frac{3\,Q}{16\,\pi\,K\,l}\;M\left(u, \frac{4\,l}{3\,r}\right) \quad \ldots\ldots\ldots\ldots\ldots (12a)$$

Also, if d = 0, a choice of z = 0 will result in

$$\overline{s} \approx \frac{Q}{4\,\pi K\,l}\;M\left(u, \frac{l}{r}\right), \quad \ldots\ldots\ldots\ldots\ldots (12b)$$

whereas, a choice of z = l will give

$$\overline{s} \approx \frac{Q}{8\,\pi K\,l}\;M\left(u, \frac{2\,l}{r}\right) \quad \ldots\ldots\ldots\ldots\ldots (12c)$$

*Equation of Average Drawdown for Relatively Large Values of Time.*—Because, for large values of time, that is for $t > \left(b^2 S_s/(2\,K)\right)$, $W\left(u, \frac{n\pi r}{b}\right)$ can be approximated very closely by $2\,K_o\left(\frac{n\pi r}{b}\right)$, Eq. 9a will, in this range of time, become

$$\overline{s} = \frac{Q}{4\,\pi K\,b}\left\{W(u) + \overline{f}_s\left(\frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{l'}{b}, \frac{d'}{b}\right)\right\} \quad \ldots\ldots\ldots (13a)$$

in which

$$\overline{f}_s = \frac{4\,b^2}{\pi^2\,(l-d)(l'-d')}\sum_{n=1}^{\infty}\frac{1}{n^2}\,K_o\left(\frac{n\,\pi\,r}{b}\right)$$
$$\left[\sin\frac{n\,\pi\,l}{b} - \sin\frac{n\,\pi\,d}{b}\right]\left[\sin\frac{n\,\pi\,l'}{b} - \sin\frac{n\,\pi\,d'}{b}\right] \quad \ldots\ldots (13b)$$

The observation made in the paragraph following Eq. 8a holds for Eq. 13a also.

## DRAWDOWN IN PIEZOMETERS OR WELLS FOR (r/b) > 1.5

For relatively large distances, that is (r/b) > 1.5, the equation of drawdown has been shown[12] to be:

$$s = \overline{s} = \frac{Q}{4\,\pi K\,b}\,W(u) \quad \ldots\ldots\ldots\ldots\ldots (14)$$

In fact, Eq. 14 gives results sufficiently accurate for practical purposes even for (r/b) as small as one, provided $u < 0.1\,(r/b)^2$. Eq. 14 is the same as it would be if the pumped well completely penetrated the aquifer (Theis formula).[13] In other words, the actual three-dimensional flowpattern changes to a radial type and hardly distinguished from that of a radial system at a dis-

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[12] "Nonsteady Flow to a Well Partially Penetrating an Infinte Leaky Aquifer," by M. S. Hantush, Proceedings, Iraq i Scientific Soc., 1957, p. 10; also reprinted by New Mexico Inst. of Mining and Tech., Socorro, N. Mex.

[13] "Groundwater Hydrology," by David K. Todd, John Wiley and Sons, Inc., New York, 1959, p. 90; also "Arid Zone Hydrology, Recent Development," by H. Schoeller, UNESCO, Paris, France, 1959, p. 37.

tance from the pumped well equal to or greater than 1.5 times the thickness of the aquifer.

## RECOVERY EQUATIONS

If t and t' are the time, reckoned respectively from the commencement and end of pumping, the residual drawdown s' in a piezometer during recovery can be shown to be

$$s' = s(t) - s(t') \qquad (15)$$

Similarly, the average residual drawdown $\bar{s}'$ is an observation well is

$$\bar{s}' = \bar{s}(t) - \bar{s}(t') \qquad (16)$$

in which $t = t_o + t'$ and $t_o$ is the time at which the pumping has ceased. Thus, the recovery equation corresponding to any of the drawdown equations discussed previously can be formulated readily, subject to the same time criteria. For example, by using Eq. 15 and Eq. 1, the general equation of recovery in a piezometer of penetration depth equal to z can be written immediately as

$$s' = \frac{Q}{4\,\pi\,K\,b}\left\{W(u) - W(u') + f\left(u, \frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}\right)\right.$$
$$\left. - f\left(u', \frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}\right)\right\} \qquad (17)$$

in which u' is the value of u after replacing t by t'.

If Eq. 2a is used instead of Eq. 1 in conjunction with Eq. 15, the general recovery equation can be put in the following alternative form:

$$s' = \frac{Q}{8\,\pi\,K(l - d)}\,\{(\text{terms of Eq. 2a})$$
$$-(\text{same terms, with } u' \text{ replacing } u)\} \qquad (18)$$

A third form can be obtained by subtracting Eq. 2a, with u' replacing u, from Eq. 1.

Whereas Eq. 17 is suitable for computation when t' is large, Eq. 18 is suitable for small values of t; that is $(t_o + t')$. The third form is suitable for computation when t is large and t' is small.

Other equations can be obtained, of course, by using drawdown equations for the different time criteria.

The preceding equations give the recovery in piezometers. The recovery in observation wells can be obtained similarly by using the appropriate average drawdown equation.

## CONCLUSIONS

A more general equation than has previously been available for the drawdown around a steady well partially penetrating an artesian aquifer of uniform thickness and infinite in areal extent has been developed. Additional parameters defining the length and space position of the water-entry face (screened section) of the pumped well, as well as of the observation wells, have been introduced. The drawdown in piezometers and the average drawdown in observation wells has been expressed analytically in forms amenable to relatively easy computation. The effect of partial penetration on the drawdown around a pumping well is shown in Figs. 2 and 3.

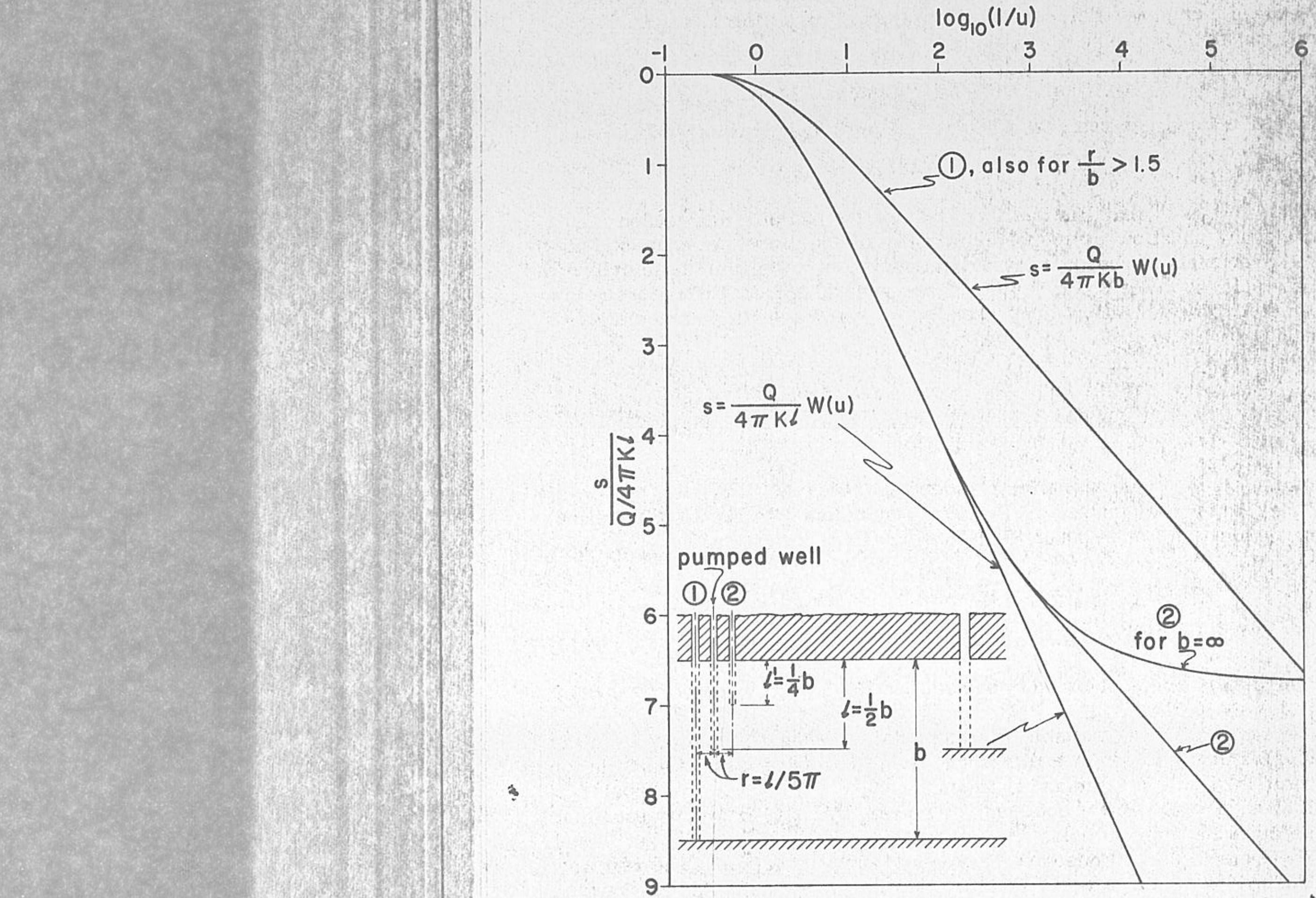

FIG. 2.—TIME DRAWDOWN VARIATION IN PARTIALLY PENETRATING WELLS

The average drawdown developed in an observation well, however close it may be to the partially penetrating pumped well, is given by the Theis formula (Eq. 14), provided that the observation well is screened throughout the aquifer. The same is true in the case of a well located at $(r/b) > 1.5$, regardless of the space position of its screen. In other words, the average drawdown in such wells is not affected by partial penetration. It is the same as though the pumped well completely penetrated the aquifer (see curve 1 of Fig. 2).

Regardless of the location of the wells and the space position of their screens, the time-drawdown curves will, at relatively large values of time $\left[t > b^2\ S_s/2\ K)\right]$, have approximately the same slope. This slope is the same as would obtain if the pumped well completely penetrated the aquifer. In other words, the effect of partial penetration has attained its maximum value (see curves 1 and 2 of Fig. 2).

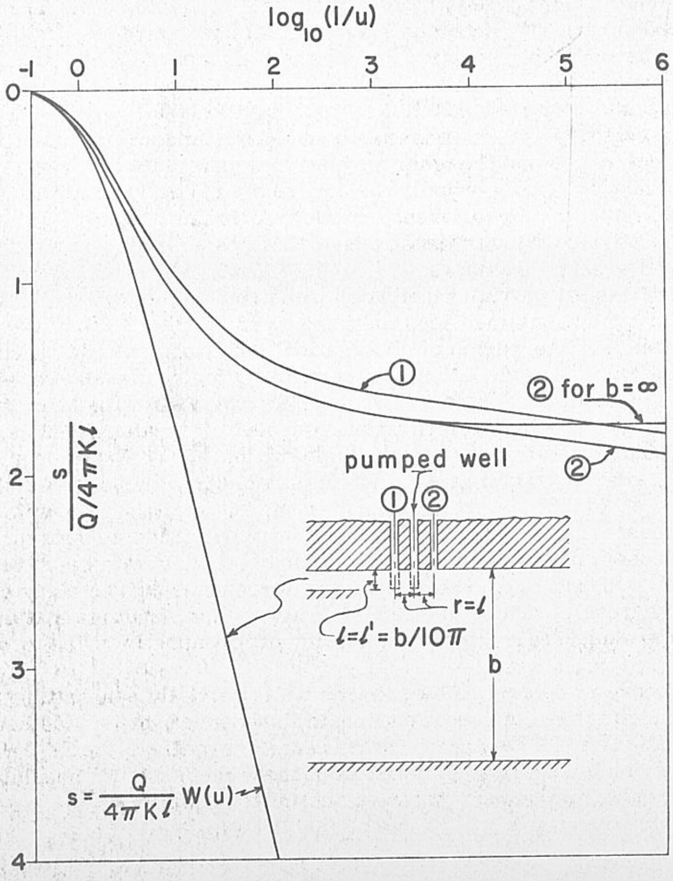

FIG. 3.—TIME-DRAWDOWN VARIATION IN WELLS OF DIFFERENT PENETRATION DEPTHS IN AN ARTESIAN AQUIFER

If the observation well is not relatively distant ($r/b < 1.5$) or is not screened throughout the aquifer, the variation of the average drawdown with the logarithm of time will have the trend shown by the curve labeled "2" in Fig. 2 and curves 1 and 2 of Fig. 3. During the early period of pumping and before the inflection of the curves appears, such time-drawdown curves have the same general appearance as curves for the case of complete penetration (Theis formula); the Theis formula, of course, is inapplicable. Even in the very early period of pumping, when one may be tempted (for the purpose of applying the Theis formula) to assume that the aquifer ends at the bottom of the pumped well, it cannot be applied except in the case in which the geometry of the flow system is such that the drawdown equation for relatively short time reduces to the form of Eq. 12b, in which case the validity of using the Theis formula is assured only in the range (see Eqs. 6 and 12b) where $t < \left(l^2 S_s/(20\ K)\right)$. However, if the drawdown equation reduces to the form of Eq. 12, the Theis formula, if modified in accordance with Eqs. 12 and 6, can be used to advantage, provided that $t < \left[(r\beta)^2 S_s/(20\ K)\right]$.

Fig. 3 compares the drawdowns observed in two equally distant wells, one of zero penetration and the other screened throughout its depth of penetration. It shows that two wells equally distant from a partially penetrating pumping well may register two different drawdowns. In fact, depending on the length and the relative position of the screens, it is possible for a more distant well to reflect a greater drawdown.

The effects of partial penetration resemble the effects of leakage from storage in a thick, semipervious confining layer.[14] Also, if the curve inflection is apparent, but the period of observation is not long enough to establish the ultimate straight line variation on a semilogarithmic time-drawdown plot, the effects of partial penetration resemble the effects of some kind of recharge boundary, such as induced infiltration from beds of streams or lakes,[15] or recharge from water-bearing strata supplying leakage[16] through semipervious confining beds. The same general effects are observed if the wells completely penetrate a sloping water-table aquifer or an aquifer of nonuniform thickness.[17] Thus, without sufficient information about a flow system that is being studied, observational drawdown trends may be interpreted in several ways. Indiscriminate use of such data may give erroneous and, in many cases, unreasonable results. This, of course, leads to the vexations that arise when attempts are made to force the application of formulas to cases to which they do not apply.

The theory of unsteady flow towards wells partially penetrating an infinite artesian aquifer will be used to outline, in a subsequent paper, methods for the determination of the formation coefficients, as well as the thickness of the water-bearing formation. Applications of these methods will be illustrated by analyzing data from ground-water basins in New Mexico.

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[14] "Modification of the Theory of Leaky Aquifers," by M. S. Hantush, Journal of Geophysical Research, Vol. 65, 1960, p. 3713.

[15] "Analysis of Data From Pumping Wells Near a River," by M. S. Hantush, Journal of Geophysical Research, Vol. 64, 1959, p. 1921.

[16] "Nonsteady Radial Flow in an Infinite Leaky Aquifer," by M. S. Hantush and C. E. Jacob, Transactions, Amer. Geophys. Union, Vol. 36, 1955, p. 95.

[17] "Ground-Water Flow in Sands of Nonuniform Thickness," by M. S. Hantush; to be published in the Journal of Geophysical Research, Vol. 66, 1961 or 1962.

$l$ = depth of penetration of the pumped well, L;

$l'$ = depth of penetration of an observation hole, L;

$M(u, \beta)$ = $\int_u^\infty \frac{e^{-y}}{y} \operatorname{erf}(\beta \sqrt{y})\, dy$, tabular values of which are given in Table 1.

$Q$ = constant well discharge, $L^3\, T^{-1}$;

$r$ = radial distance measured from center of well, L;

$s$ = drawdown of piezometric surface at any time and at any point in the aquifer (drawdown in piezometers), L;

$\overline{s}$ = average drawdown in observation holes, L;

$s'$ = residual drawdown in piezometers, L;

$\overline{s}'$ = residual drawdown in observation holes, L;

$S$ = $b\, S_s$ = storage coefficient;

$S_s$ = specific storage (volume of water released from storage by a unit volume of the aquifer under a unit head decline), $L^{-1}$;

$t$ = time since pumping started, T;

$t_o$ = period of pumping, T;

$t'$ = time since pumping stopped, T;

$T$ = $K\, b$ = transmissivity of the aquifer, $L^2\, T^{-1}$;

$u$ = $\left(r^2 S_s / (4\, K\, t)\right)$;

$u'$ = $\left(r^2\, S_s / (4\, K\, t')\right)$;

$W(u)$ = $\int_u^\infty \frac{e^{-y}}{y}\, dy$ = well function of $(u)$ for nonleaky aquifers for which tables are available;

$W(u,x)$ = $\int_u^\infty \frac{dy}{y} \exp\left(-y - x^2/(4\, y)\right)$ = well function of ($u$ and $x$) for leaky aquifers for which tables are available; and

$z$ = vertical coordinate measured from the top of the aquifer, positive downward.