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Effect of Partial Penetration on Flow in Unconfined Aquifers Considering Delayed Gravity Response

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Previously, a new analytical model was proposed by the author for the delayed response process characterizing flow to a well in an unconfined aquifer. It was shown that contrary to prevailing belief this process can be simulated by using constant values of specific storage and specific yield without recourse to unsaturated flow theory. In the present work the theory is extended to account for the effect of a well partially penetrating a homogeneous anisotropic unconfined aquifer. Field and laboratory evidence is quoted to suggest that the elastic storage properties of unconfined aquifers may often be much more pronounced than those of deep-seated confined aquifers composed of similar materials. The need for further research into the mechanical properties of unconfined aquifers and the role of unsaturated flow in the response process is emphasized.

The phenomenon commonly known as 'delayed yield' that characterizes flow to wells in unconfined aquifers has often been attributed to unsaturated flow above the water table. Recently, *Neuman* [1972*a*, 1973] showed that this phenomenon can be simulated mathematically by using constant values of specific storage and specific yield without recourse to unsaturated flow theory. The new model treats the unconfined aquifer as a compressible system and the phreatic surface as a moving material boundary. It differs from that of *Boulton* [1954, 1963, 1970] and *Boulton and Pontin* [1971] in that it is based only on well-defined physical parameters of the aquifer system. Results from this model suggest that in the absence of vertical recharge or plant uptake near the water table, compressibility may often be much more important than unsaturated flow in determining the rate of drawdown in the saturated zone of the aquifer.

Concurrently with our work, *Streltsova* [1972*a*, *b*], partly in collaboration with Rushton [*Streltsova and Rushton*, 1973], was able to develop approximate solutions for the fall of the water table, as well as for the average drawdown over the entire aquifer thickness, in response to a fully penetrating well discharging at a constant rate. Her model has some conceptual similarities to ours because the unsaturated zone is neglected and water is released from storage only by compaction of the aquifer material, expansion of the water, and gravity drainage at the phreatic surface. In a later work, *Streltsova* [1973] relied on a field experiment by *Meyer* [1962] to demonstrate the subordinate role that unsaturated flow plays in the delayed yield process. In addition, she showed that *Boulton*'s [1954, 1963] classical convolution integral can be derived without considering the unsaturated zone merely by taking into account vertical gradients underneath the moving water table. *Cooley and Case* [1973] showed independently that this convolution integral can also be viewed as the vertical recharge into an aquifer overlain by a rigid water table aquitard when unsaturated flow above the phreatic surface is neglected. They compared the rate of recharge with and without the presence of an unsaturated zone and found that the lack of an unsaturated zone has little influence on the results. In light of these findings as well as those of *Brutsaert et al.* [1971] we tend to adopt *Streltsova*'s [1973, p. 230] statement that

> The unique linear relationship between the unsteady water table heights at any time and the flux at the water table averaged over the area, which has repeatedly been confirmed by experimental results of *Childs* [1960] and *Dos Santos and Youngs* [1969], is . . . actually proof of the constant value of the specific yield. It is the free surface and the flux that have an exponential form of change in time . . . and not the coefficient S. . . .

(In this paper, S_y is used instead of S.)

Another argument concerns the importance of the elastic storage properties of unconfined aquifers due to compressibility. Here we again recall *Walton*'s [1960, p. 46] words, based on field experience, that

> Unconfined stratified sediments often react to pumping for a short time after pumping begins as would an artesian aquifer. Gravity drainage is not immediate but water is released instantaneously from storage by the compaction of the aquifer and its associated beds and by the expansion of the water itself.

Later in our discussion we shall present field and laboratory evidence that suggests that unconfined aquifers may often be much more compressible than comparable deep-seated confined materials.

In this paper the model of *Neuman* [1972*a*, 1973] is extended to account for the effect of partial penetration on drawdowns in an unconfined aquifer. The extended model takes into account aquifer anisotropy, and it includes *Hantush*'s [1964] solution for a compressible confined aquifer and *Dagan*'s [1967*a*, *b*] solution for a rigid water table aquifer as particular cases. When the pumping well fully penetrates the aquifer, the new solution collapses to that of *Neuman* [1972*a*, 1973]. Since the results apply to both rising and falling water tables, the term 'delayed gravity response' is preferred over the classical delayed yield concept.

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THEORETICAL DEVELOPMENT

Consider an unconfined aquifer of infinite lateral extent that rests on an impermeable horizontal layer such as that shown schematically in Figure 1. The aquifer material is uniform and anisotropic, the principal permeabilities being oriented parallel to the coordinate axes. A well discharging at a constant rate Q is open to inflow from a depth d to a depth l beneath the initially static water table. It is assumed that water is released from storage by compaction of the aquifer material, expansion of the water, and gravity drainage at the free surface.

In the analytical approach it is convenient to treat the well as a line sink, i.e., to neglect well storage as well as the presence of a seepage face. This introduces a certain error in the vicinity of the well bore, as has been pointed out by *Kipp* [1973], who has solved this problem for a finite well in a rigid semiinfinite aquifer by neglecting the seepage face. The governing equations become [*Neuman*, 1972*a*]

$$K_r \frac{\partial^2 s}{\partial r^2} + \frac{K_r}{r}\frac{\partial s}{\partial r} + K_z \frac{\partial^2 s}{\partial z^2} = S_s \frac{\partial s}{\partial t} \qquad 0 < z < \xi \tag{1}$$

$$s(r, z, 0) = 0 \tag{2}$$

$$\xi(r, 0) = b \tag{3}$$

$$s(\infty, z, t) = 0 \tag{4}$$

$$\partial s/\partial z(r, 0, t) = 0 \tag{5}$$

$$K_r \frac{\partial s}{\partial r} n_r + K_z \frac{\partial s}{\partial z} n_z = \left(S_y \frac{\partial \xi}{\partial t} - I\right) n_z \quad \text{at} \quad z = \xi \tag{6}$$

$$\xi(r, t) = b - s(r, \xi, t) \tag{7}$$

$$\frac{\partial s}{\partial z}(0, z, t) = 0 \text{ at } 0 \le z \le b - l \quad b - d \le z \le \xi \tag{8}$$

$$\lim_{r \to 0} \int_{b-l}^{\min(b-d, \xi)} r \frac{\partial s}{\partial r}\, dz = -\frac{Q}{2\pi K_r} \tag{9}$$

Equations 1–9 can be linearized by using a perturbation technique similar to that employed by *Dagan* [1967*a*, *b*] provided that the aquifer is thick enough and s remains much smaller than ξ. Here this technique leads to a first-order linearized approximation, obtained simply by shifting the boundary condition from the free surface to the horizontal plane $z = b$. This eliminates ξ from (1)–(9), and in the absence of recharge or plant uptake at the free surface one obtains

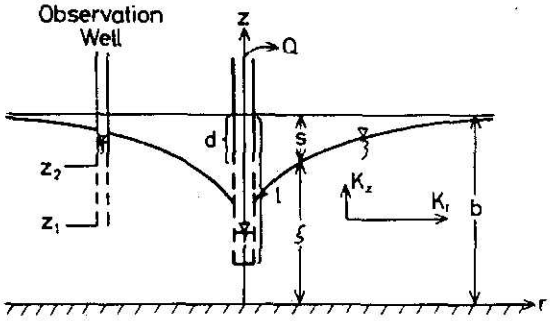

Fig. 1. Schematic diagram of unconfined aquifer.

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + K_D \frac{\partial^2 s}{\partial z^2} = \frac{1}{\alpha_s}\frac{\partial s}{\partial t} \qquad 0 < z < b \tag{10}$$

$$s(r, z, 0) = 0 \tag{11}$$

$$s(\infty, z, t) = 0 \tag{12}$$

$$\partial s/\partial z(r, 0, t) = 0 \tag{13}$$

$$\frac{\partial s}{\partial z}(r, b, t) = -\frac{1}{\alpha_y}\frac{\partial s}{\partial t}(r, b, t) \tag{14}$$

$$\frac{\partial s}{\partial z}(0, z, t) = 0 \text{ at } 0 \le z \le b - l \quad b - d \le z \le b \tag{15}$$

A further approximation is introduced by assuming that flux along the perforated section of the well is uniform, so that (9) becomes

$$\lim_{r \to 0} r \frac{\partial s}{\partial r} = -\frac{Q}{2\pi K_r (l - d)} \text{ at } b - l < z < b - d \tag{16}$$

After applying the Laplace and Hankel transforms to (10)–(16) and inverting the results, one obtains a first-order approximation to the original initial boundary value problem. The mathematics involved are outlined in Appendix 1. The final solution is expressed in terms of six dimensionless parameters: σ, β, z_D, l_D, d_D, and t_s or t_y. The solution has the form

$$s(r, z, t) = \frac{Q}{4\pi T}\int_0^\infty 4y J_0(y\beta^{1/2})\left[u_0(y) + \sum_{n=1}^{\infty} u_n(y)\right] dy \tag{17}$$

where

$$u_0(y) = \frac{\{1 - \exp[-t_s\beta(y^2 - \gamma_0^2)]\}\cos(\gamma_0 z_D)}{[y^2 + (1+\sigma)\gamma_0^2 - (y^2 - \gamma_0^2)^2/\sigma]\cos(\gamma_0)} \cdot \frac{\sinh[\gamma_0(1 - d_D)] - \sinh[\gamma_0(1 - l_D)]}{(l_D - d_D)\sinh(\gamma_0)}$$

$$u_n(y) = \frac{\{1 - \exp[-t_s\beta(y^2 + \gamma_n^2)]\}\cos(\gamma_n z_D)}{[y^2 - (1+\sigma)\gamma_n^2 - (y^2 + \gamma_n^2)^2/\sigma]\cos(\gamma_n)} \cdot \frac{\sin[\gamma_n(1 - d_D)] - \sin[\gamma_n(1 - l_D)]}{(l_D - d_D)\sin(\gamma_n)}$$

and the terms γ_0 and γ_n are the roots of the equations

$$\sigma\gamma_0 \sinh(\gamma_0) - (y^2 - \gamma_0^2)\cosh(\gamma_0) = 0 \quad \gamma_0^2 < y^2 \tag{18}$$

$$\sigma\gamma_n \sin(\gamma_n) + (y^2 + \gamma_n^2)\cos(\gamma_n) = 0 \tag{19}$$

$$(2n - 1)(\pi/2) < \gamma_n < n\pi \qquad n \ge 1$$

Since (17) is the solution to a linear problem, one can use the principle of superposition to obtain solutions for any number of pumping or injecting wells in the aquifer. In the case of a fully penetrating well, (17) reduces to (1) of *Neuman* [1973].

AVERAGE DRAWDOWN IN OBSERVATION WELL

The drawdown recorded in an observation well that is perforated between the elevations z_1 and z_2 (Figure 1) is simply the average over that vertical distance and is given by

$$s_{z_1 \cdots z_2}(r, t) = (z_2 - z_1)^{-1}\int_{z_1}^{z_2} s(r, z, t)\, dz \tag{20}$$

This average drawdown is always greater than the fall of

the water table at the point of observation. It is easy to verify that s_{z_1,z_2} can be calculated directly from (17) by merely redefining the expressions

$$u_0(y) = \frac{\{1 - \exp[-t_s\beta(y^2 - \gamma_0^2)]\}\{\sinh(\gamma_0 z_{2D}) - \sinh(\gamma_0 z_{1D})\}\{\sinh[\gamma_0(1 - d_D)] - \sinh[\gamma_0(1 - l_D)]\}}{[y^2 + (1 + \sigma)\gamma_0^2 - (y^2 - \gamma_0^2)^2/\sigma]\cosh(\gamma_0)\cdot(z_{2D} - z_{1D})\gamma_0(l_D - d_D)\sinh(\gamma_0)} \tag{21}$$

$$u_n(y) = \frac{\{1 - \exp[-t_s\beta(y^2 + \gamma_n^2)]\}\{\sin(\gamma_n z_{2D}) - \sin(\gamma_n z_{1D})\}\{\sin[\gamma_n(1 - d_D)] - \sin[\gamma_n(1 - l_D)]\}}{[y^2 - (1 + \sigma)\gamma_n^2 - (y^2 + \gamma_n^2)^2/\sigma]\cos(\gamma_n)\cdot(z_{2D} - z_{1D})\gamma_n(l_D - d_D)\sin(\gamma_n)} \tag{22}$$

Reduction to Hantush's Solution for a Confined Aquifer

If there is no gravity drainage ($S_y = 0$ and $\sigma = \infty$), the aquifer is artesian, and (17) should reduce to *Hantush's* [1964] solution. From (18) one has

$$\lim_{\sigma\to\infty} \gamma_0 = 0$$

and when it is noted that

$$\lim_{\gamma_0\to 0} \frac{\sinh[\gamma_0(1 - d_D)] - \sinh[\gamma_0(1 - l_D)]}{(l_D - d_D)\sinh(\gamma_0)} = 1$$

it becomes obvious that

$$\lim_{\sigma\to\infty} u_0(y) = \lim_{\sigma\to\infty} w_0(y)$$

where $w_0(y)$ is the equivalent of $u_0(y)$ for complete penetration (see (1) of *Neuman* [1973]). This together with (22)–(26) of Neuman [1972*a*] shows that the first part of (17) reduces to

$$\lim_{\sigma\to\infty} \frac{Q}{4\pi T}\int_0^\infty 4y J_0(y\beta^{1/2})u_0(y)\,dy = \frac{Q}{4\pi T}\int_{1/4t_s}^\infty \frac{e^{-y}}{y}\,dy \tag{23}$$

which is the Theis solution with respect to t_s.

On the other hand, from (19) one has

$$\lim_{\sigma\to\infty} \gamma_n = n\pi \qquad n > 1$$

and

$$\lim_{\sigma\to\infty} \sigma\gamma_n^2 \sin(\gamma_n) = \lim_{\gamma_n\to n\pi} -(y^2 + \gamma_n^2)\gamma_n\cos(\gamma_n)$$
$$= -(y^2 + n^2\pi^2)n\pi\cos(n\pi)$$

which means that

$$\lim_{\sigma\to\infty} u_n(y) = \frac{\{1 - \exp[-t_s\beta(y^2 + n^2\pi^2)]\}\cos(n\pi z_D)\{\sin[n\pi(1 - d_D)] - \sin[n\pi(1 - l_D)]\}}{(y^2 + n^2\pi^2)n\pi(l_D - d_D)}$$

This together with (C8) in Appendix 3 shows that the second part of (17) reduces to

$$\lim_{\sigma\to\infty} \frac{Q}{4\pi T}\int_0^\infty 4y J_0(y\beta^{1/2})\sum_{n=1}^\infty u_n(y)\,dy$$
$$= \frac{Q}{4\pi T}\frac{2}{\pi(l_D - d_D)}\sum_{n=1}^\infty \frac{1}{n}\cos[n\pi(1 - z_D)]$$
$$\cdot[\sin(n\pi l_D) - \sin(n\pi d_D)]$$
$$\cdot\int_{1/4t_s}^\infty \exp\left(-y - \frac{\beta n^2\pi^2}{4y}\right)\frac{dy}{y} \tag{24}$$

The sum of (23) and (24) is *Hantush's* [1964, p. 350, equation 74] solution for partial penetration in a confined aquifer without leakage.

Asymptotic Behavior at Large Values of Dimensionless Time

At large values of dimensionless time t_s the solution behaves as if the aquifer were rigid and S_s were zero ($S_s = 0$ corresponds to $t_s = \infty$). This asymptotic behavior is represented by *Dagan's* [1967*a*, *b*] solution for partial penetration in an incompressible unconfined aquifer. Since (17) cannot be reduced analytically to Dagan's solution, the correspondence between these two solutions at large values of dimensionless time will be demonstrated numerically below.

Evaluation of Analytical Solutions

In order to investigate the effect of partial penetration on flow in an elastic unconfined aquifer, (17) and (20) have been evaluated numerically. The numerical approach is similar to that described in Appendix D in *Neuman* [1972*a*]. A listing of the computer program complete with users' instructions is available from the author on request.

It is instructive to start our discussion by comparing the present delayed response solution with the solutions of *Hantush* [1964] for a confined elastic aquifer and *Dagan* [1967*a*, *b*] for an unconfined rigid aquifer. For this purpose, let us consider a situation in which the pumping well penetrates to a depth of $0.2b$ below the initial water table ($l_D = 0.2$) in an aquifer characterized by $\sigma = S/S_y = 10^{-2}$. Figure 2 shows the dimensionless time-drawdown curves obtained by all three solutions for a fixed point in the aquifer, situated at a radial distance $r = 0.6b/K_0^{1/2}$ from the center of the pumping well ($\beta = 0.36$) at an elevation $z = 0.85b$ above the bottom of the layer ($z_D = 0.85$). The sketch in Figure 2 gives a visual description of this situation for the particular case in which the aquifer is isotropic, so that $K_D = 1$ and $r = 0.6b$. The solutions of Hantush and Dagan were evaluated from tables published by *Witherspoon et al.* [1967] and *Dagan* [1967*b*].

The time-drawdown curve in Figure 2 suggests that water is released from storage in three stages, as discussed by *Walton* [1960]. At early values of time the curve approaches Hantush's solution and thus indicates that water is released from storage primarily by compaction of the aquifer material and expansion of the water. During the second stage, gravity drainage becomes important, and its effect is similar to that of leakage from a nearby source. We showed previously [*Neuman*, 1972*a*] that the smaller

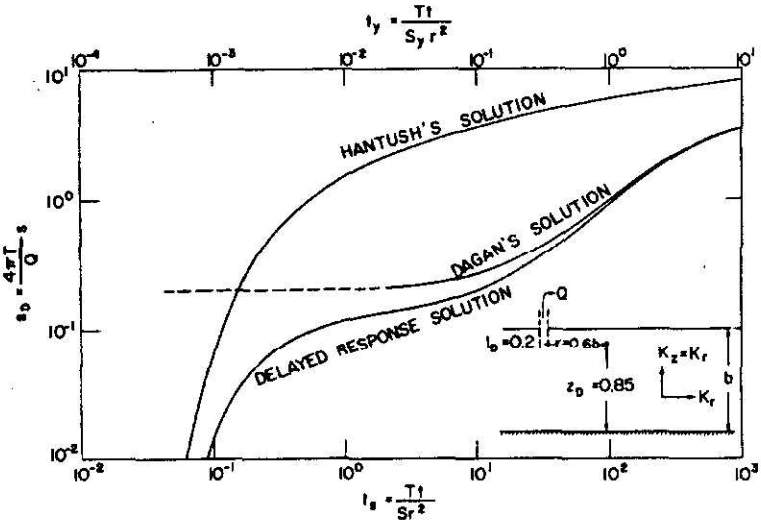

Fig. 2. Comparison of delayed response solution with the solutions of Hantush and Dagan for $\sigma = 10^{-2}$, $\beta = 0.36$, $l_D = 0.2$, $d_D = 0$, and $z_D = 0.85$.

σ is, the larger the effect of gravity drainage is and therefore the more pronounced this 'leakage' is.

As time increases, the effect of elastic storage at the point considered dissipates completely, and the time-drawdown curve approaches Dagan's solution asymptotically. Note that Dagan's solution is expressed in terms of t_y, whereas Hantush's solution is expressed in terms of t_s. The delayed response solution, on the other hand, can be expressed in terms of either t_s or t_y, since these two parameters are related by $t_y = \sigma t_s$. Both scales are indicated in Figure 2.

We know [see *Neuman*, 1972*a*] that when the pumping well completely penetrates the aquifer, the time-drawdown curves are bounded by the Theis solution with respect to t_s during the early stage and by the Theis solution with respect to t_y during the late gravity drainage stage. We now see that in the case of partial penetration, Hantush's solution becomes the envelope at early time and Dagan's solution becomes the envelope at later times. It is important to recognize that when the aquifer is considered to be rigid, the governing equation becomes elliptic and therefore Dagan's solution is discontinuous in time at $t = 0$. In fact, this solution implies that water levels at all points below the phreatic surface must drop instantaneously when pumping starts. In Figure 2, for example, the dimensionless drawdown according to Dagan changes instantaneously from zero to somewhere between 0.1 and 0.2 at time $t = 0$, and its value remains nearly constant until t_y exceeds 10^{-2}. The behavior of Dagan's solution at $t_y < 10^{-2}$ is indicated approximately by the broken curve in Figure 2.

We recall from *Neuman* [1972*a*] that the period of time occupied by the early segment of the time-drawdown curve at a fixed point in the aquifer becomes less as σ decreases. When σ approaches zero, this segment disappears completely, and hydraulic heads everywhere below the water table drop instantaneously when pumping starts. Thus Dagan's solution becomes more representative of the actual behavior of the aquifer as σ decreases, but it becomes more in error as σ increases.

In the literature there is an increasing amount of evidence to support the view that the elastic properties of shallow sandy aquifers may be too important to be neglected. *Prickett* [1965], after having conducted 18 pumping tests in the vicinity of Lawrenceville, Illinois, found that σ had exceeded 10^{-2} in 14 out of the 18 different test locations. More recently, *Huang* [1973] performed a laboratory pumping experiment on a confined sandy aquifer model 40 inches thick and discovered that the elastic storage coefficient S was 7.04×10^{-3}. This is equivalent to a specific storage S_s of 2.11×10^{-3} ft^{-1}, which is greater by 2–4 orders of magnitude than numbers usually reported for confined sandy aquifer materials. Huang explained that this difference was '. . . probably due to the loose nature of the sand and the small overburden pressure employed in the test.' *Vachaud et al.* [1971] conducted laboratory drainage experiments on an unconfined sandy aquifer model 2 m high, 3 m long, and 5 cm wide with the water table initially at an elevation of 143 cm above the bottom of the sand. In an attempt to reproduce their results numerically by the finite element method we found [*Neuman*, 1972*b*] that the only way to accomplish this was by assigning a specific storage value to the sand of the order of 1.5×10^{-2} ft^{-1}. This value is even higher than that reported by Huang and might possibly be due to the loosening of the sand during wetting and the lack of any significant overburden pressure during the tests. On the basis of such information we are tempted to assert that contrary to prevailing belief the elastic storage properties of unconfined aquifers may be considerably more pronounced than those of corresponding deep-seated confined materials. Additional research is needed to verify this assertion. However, if it is true, then elastic storage must generally be taken into account in dealing with unconfined aquifers.

Figure 3 shows how the dimensionless drawdown varies with elevation and with radial distance from the pumping well at $t_s = 1/\beta$ when the aquifer is isotropic, $\sigma = 10^{-2}$, and the well penetrates two tenths of the upper original saturated thickness. Note that at distances exceeding those

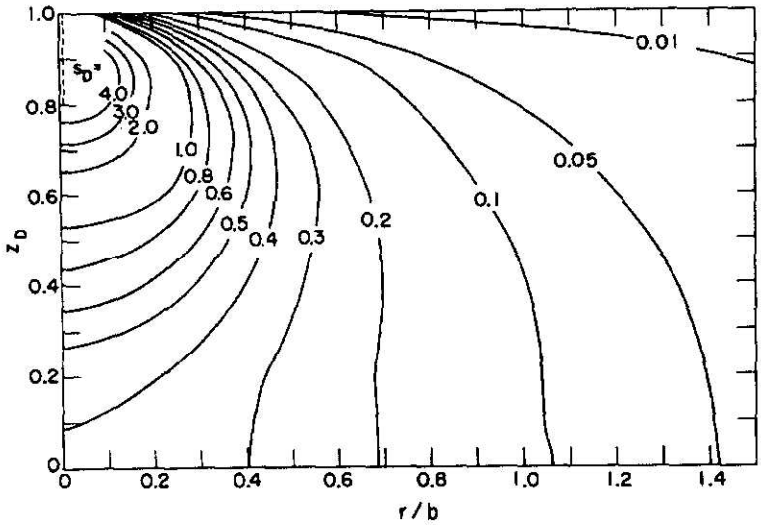

Fig. 3. Dimensionless flow pattern around pumping well for $\sigma = 10^{-2}$, $\beta = r^2/b^2$, $t_s\beta = 1.0$, $l_D = 0.2$, and $d_D = 0$.

corresponding to $\beta = 1$ the drawdown at the water table is less than the drawdown at greater depths. Thus at dimensionless times less than $t_s = 1$ the response of the water table lags behind the drawdown in the rest of the aquifer, and hence the terms delayed yield, delayed gravity response, or delayed response of the water table are derived.

Figure 4 shows how the depth of penetration affects the drawdown at the bottom of the aquifer ($z_D = 0$) when $\sigma = 10^{-2}$ and $\beta = 10^{-3}$. The sketch in Figure 4 depicts a particular case in which the horizontal permeability is 10 times the vertical permeability and the radial distance is $r = 0.1b$. The Theis solution for t_s and Hantush's solution as obtained from tables published by *Witherspoon et al.* [1967] are included for reference purposes. Data about Hantush's solution at $l_D = 0.2$ and $z_D = 0$ are not available in these tables and are therefore omitted from Figure 4.

It is seen that as the well penetrates nearer to the point of observation, an earlier response is observed and the drawdown is greater. Owing to the small value of β (which can mean either that the observation point is radially very close to the well or that the horizontal permeability is much greater than the vertical one), the early stage of elastic response is highly pronounced, and the time-drawdown curves coincide with the Theis solution when $l_D = 1$ or with Hantush's solution when $l_D < 1$ during this stage. The less the depth of penetration is, the earlier the drawdown deviates from these solutions, an indication that the importance of gravity effects increases as the depth of penetration decreases. This increase is of course expected because gravity is associated with the vertical component of the flow velocity vector, which becomes more important when the depth of penetration is small.

The effect of β on drawdown in the aquifer at $z_D = 0$, when the well either completely or partially penetrates the saturated zone, is illustrated in Figure 5 for $\sigma = 10^{-2}$. The sketch in Figure 5 gives a visual concept of these situations for the particular case in which the aquifer is isotropic and $\beta^{1/2}$ is directly proportional to the radial

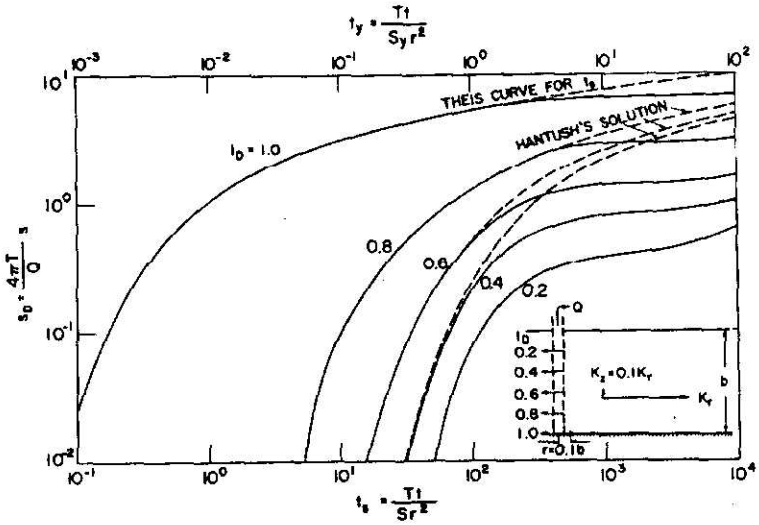

Fig. 4. Effect of penetration depth on time-drawdown curves for $\sigma = 10^{-2}$, $\beta = 10^{-3}$, $d_D = 0$, and $z_D = 0$.

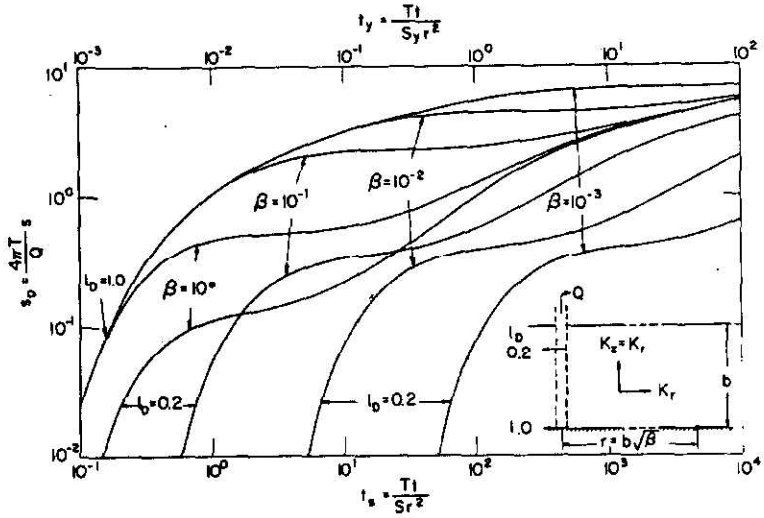

Fig. 5. Effect of β on time-drawdown curves for $\sigma = 10^{-2}$, $l_D = 0.2$ and 1.0, $d_D = 0$, and $z_D = 0$.

distance from the pumping well. One notes that the greater the value of β is, the less the effect of partial penetration on the drawdown is. The effect of partial penetration decreases with time at an increasing rate as β becomes larger. When $\beta \geq 1$, this effect disappears completely at dimensionless time values $t_y \geq 10$. This disappearance means that flow at radial distances exceeding $r = b\beta^{1/2}/K_D^{1/2}$ becomes essentially horizontal whereas the water is released from storage only by gravity drainage. These results are consistent with the well-known Dupuit assumptions. Figure 5 also indicates that the effect of elastic storage decreases with increasing values of β, as one would intuitively anticipate.

If instead of drawdown at a point one considers the average drawdown in an observation well that is perforated throughout the entire saturated depth of the aquifer, the results will be as those shown in Figure 6. A comparison of Figures 5 and 6 will demonstrate that partial penetration has a much smaller effect on the average drawdown than on the drawdown at a point. Thus the time-drawdown curves are nearer to the Theis solution with respect to t_s at early time and to the Theis solution with respect to t_y at later times, as previously indicated by *Neuman* [1972*a*]. Both Theis solutions are included in Figure 6 for reference purposes. It appears that the effect of partial penetration on the average drawdown can be neglected for all values of $\beta \geq 1$ whenever $t_y \geq 1$.

CONCLUSIONS

The process of delayed gravity yield in response to pumping from a well partially penetrating a homogeneous anisotropic unconfined aquifer can be simulated by using constant values of specific storage S_s and specific yield S_y. A simple mathematical model of the delayed yield process is obtained without considering the unsaturated zone merely by treating the unconfined aquifer as a compressible system with a moving material boundary at the water table.

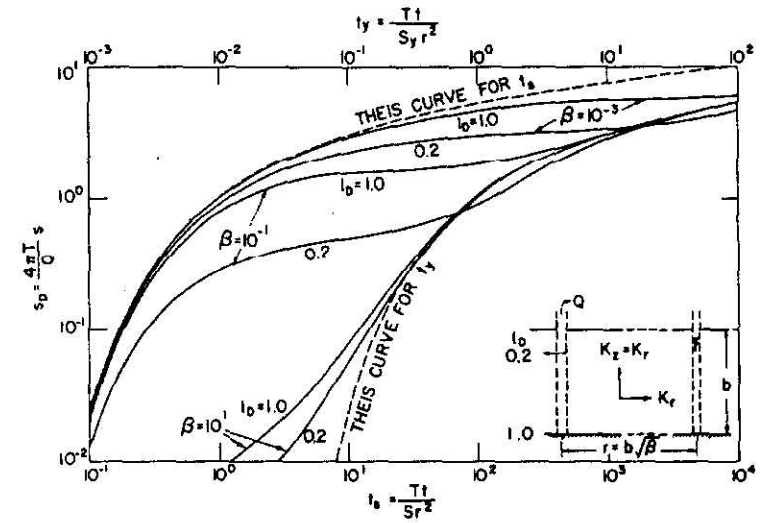

Fig. 6. Effect of β on depth-averaged time-drawdown curves for $\sigma = 10^{-2}$, $l_D = 0.2$ and 1.0, $d_D = 0$, and $z_D = 0$.

There is an increasing amount of evidence in the literature to support the view that unconfined aquifers may often be much more compressible than comparable deep-seated confined aquifers. On the other hand, unsaturated flow above the water table appears to have little effect on the drawdown provided that the water table is below the root zone and the aquifer is not too thin. This suggests that one may often be committing a much greater error in neglecting the elastic properties of unconfined aquifers than in disregarding the unsaturated zone. Since available field data are scarce, much additional research is needed about the mechanical properties of unconfined aquifers and about the effect of unsaturated flow on drawdowns in the saturated zone before this conclusion can be properly qualified.

The effect of partial penetration on drawdown in an unconfined aquifer decreases with radial distance from the pumping well and with the ratio K_z/K_r. At distances greater than $r = b/K_D^{1/2}$ this effect disappears completely when time exceeds $t = 10S_yr^2/T$. The influence of partial penetration can be minimized by perforating the observation well throughout the entire saturated thickness of the aquifer. In such a case the drawdown at distances exceeding $r = b/K_D^{1/2}$ will follow the Theis solution with respect to t_y at times greater than $t = S_yr^2/T$.

APPENDIX 1

To solve the initial boundary value problem posed by (10)–(16), it is convenient to divide s into two components,

$$s(r, z, t) = s_1(r, z, t) + s_2(r, z, t) \tag{A1}$$

Although both s_1 and s_2 satisfy (10)–(13) and (15), there is a change in boundary conditions (14) and (16), which now take the form

$$\partial s_1/\partial z\,(r, b, t) = 0 \tag{A2}$$

$$\frac{\partial s_2}{\partial z}(r, b, t) = -\frac{1}{\alpha_y}\frac{\partial(s_1 + s_2)}{\partial t}(r, b, t) \tag{A3}$$

$$\lim_{r\to 0} r\frac{\partial s_1}{\partial r} = -\frac{Q}{2\pi K_r(l - d)} \quad \text{at} \quad b - l < z < b - d \tag{A4}$$

$$\lim_{r\to 0} r\frac{\partial s_2}{\partial r} = 0 \quad \text{at} \quad b - l < z < b - d \tag{A5}$$

Hantush [1964, pp. 347–350] showed that s_1 is the sum of two components,

$$s_1 = s_0 + \Delta s_0 \tag{A6}$$

where s_0 represents the Theis equation with respect to t_y,

$$s_0 = \frac{Q}{4\pi T}\int_{1/4t_y}^{\infty}\frac{e^{-y}}{y}\,dy \tag{A7}$$

and Δs_0 is a correction term due to partial penetration, given by

$$\Delta s_0 = \frac{Q}{4\pi T}\frac{2}{\pi(l_D - d_D)}\sum_{n=1}^{\infty}\frac{1}{n}\cos[n\pi(1 - z_D)] \cdot [\sin(n\pi l_D) - \sin(n\pi d_D)]\int_{1/4t_y}^{\infty}\exp\left[-y - \frac{\beta n^2\pi^2}{4y}\right]\frac{dy}{y} \tag{A8}$$

When first Laplace and then Hankel transforms are applied to (A8) and when (C1) and (C4) (Appendix 3) are used, one obtains the double transform

$$\Delta s_0^* = \frac{Q}{2\pi TK_Dp}\frac{2}{\pi(l_D - d_D)}\sum_{n=1}^{\infty}\frac{1}{n}\cos[n\pi(1 - z_D)] \cdot [\sin(n\pi l_D) - \sin(n\pi d_D)]\frac{1}{\eta^2 + n^2\pi^2/b^2} \tag{A9}$$

where $\eta^2 = (a^2/K_D) + (p/\alpha_yK_D)$. A similar procedure with respect to (A7) leads to

$$s_0^* = Q/2\pi TK_Dp\eta^2 \tag{A10}$$

When the Laplace and Hankel transforms are applied to (10)–(12), (A3), and (A5) with respect to s_2 and the particular Hankel transform given by (B3) (Appendix 2) is used, the result is an ordinary differential equation in terms of z. When this differential equation is solved by using (13) and the result is summed with (A9) and (A10), one obtains the double transform of s,

$$s^* = (Q/2\pi TK_Dp\eta^2) + c\cosh(\eta z) + \Delta s_0^* \tag{A11}$$

The constant c is determined from boundary conditions (A2) and (A3) by first transforming these into

$$\partial s^*/\partial z(a, b, p) = -(p/\alpha_y)s^*(a, b, p) \tag{A12}$$

and then substituting (A11) into (A12). The result is

$$c\left[\frac{\alpha_y\eta\sinh(\eta b)}{p} + \cosh(\eta b)\right] = -\frac{Q}{2\pi TK_Dp}\left(\frac{1}{\eta^2} + f\right), \tag{A13}$$

where

$$f = \frac{2}{\pi(l_D - d_D)}\sum_{n=1}^{\infty}\frac{\sin(n\pi l_D) - \sin(n\pi d_D)}{n(\eta^2 + n^2\pi^2/b^2)}$$

The term f can be rewritten in the form

$$f = \frac{2}{\pi\eta^2(l_D - d_D)}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{n}{\eta^2 + b^2\eta^2/\pi^2}\right) \cdot [\sin(n\pi l_D) - \sin(n\pi d_D)]$$

From tables of Fourier transforms [*Churchill*, 1958, p. 299] one finds that this is equivalent to

$$f = -\frac{1}{\eta^2} - \frac{\sinh[b\eta(1 - l_D)] - \sinh[b\eta(1 - d_D)]}{\eta^2(l_D - d_D)\sinh(b\eta)}$$

When f is substituted into (A13), c is solved for, and the result is inserted into (A11), the double transform of s finally becomes

$$s^* = \frac{Q}{2\pi TK_Dp}\left(\frac{1}{\eta^2} + \frac{p\{\sinh[b\eta(1 - l_D)] - \sinh[b\eta(1 - d_D)]\}\cosh(\eta z)}{(l_D - d_D)\sinh(b\eta)[\alpha_y\eta\sinh(\eta b) + p\cosh(\eta b)]}\right) + \Delta s^* \tag{A14}$$

APPENDIX 2

The infinite Hankel transform of order zero $F(a)$ of a function $f(r)$ is defined as

$$H[f(r)] = F(a) = \int_0^\infty r J_0(ar) f(r)\, dr \qquad \text{(B1)}$$

The inverse of this transform is given by

$$H^{-1}[F(a)] = f(r) = \int_0^\infty a J_0(ra) F(a)\, da \qquad \text{(B2)}$$

The particular relationships that are used in this work are

$$H\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\langle s\rangle(r, z, p)\right]\right\}$$

$$= -a^2 s^*(a, z, p) - \lim_{r\to 0} r\frac{\partial}{\partial r}\langle s\rangle(r, z, p) \qquad \text{(B3)}$$

and

$$H^{-1}[a^{2n}/(a^2 + \xi^2)] = (-1)^n \xi^{2n} K_0(r\xi) \qquad \text{(B4)}$$

where ξ is a real constant and n is a nonnegative integer. The details of the development have been shown elsewhere [*Neuman and Witherspoon*, 1969].

APPENDIX 3

Let $\langle f\rangle(r, p)$ be the Laplace transform of some function $f(r, t)$, and let $f^*(a, p)$ be the Hankel transform of $\langle f\rangle$. Consider the particular double transform

$$f^* = [p(a^2 + \xi^2)]^{-1} \qquad \text{(C1)}$$

where $\xi^2 = (p/\alpha_r) + (K_D m^2\pi^2/b^2)$. According to (B4) the inverse Hankel transform of (C1) is

$$\langle f\rangle = K_0(r\xi)/p \qquad \text{(C2)}$$

From tables of definite integrals [*Gradshtein and Ryzhik*, 1965, p. 340] one finds that (C2) can be rewritten as

$$\langle f\rangle = \frac{1}{2}\int_0^\infty \exp\left(-y\frac{\beta m^2\pi^2}{4y}\right)\langle g\rangle(p)\frac{dy}{y} \qquad \text{(C3)}$$

where $\langle g\rangle(p) = (1/p)\exp(-pr^2/4\alpha_r y)$. When tables of Laplace transforms [*Churchill*, 1958, p. 327] are used, the inverse of (C3) becomes

$$f = \frac{1}{2}\int_{1/4t_r}^\infty \exp\left(-y - \frac{\beta m^2\pi^2}{4y}\right)\frac{dy}{y} \qquad \text{(C4)}$$

Another way to obtain the inverse Hankel transform of (C1) is by using (B2), which after a change of variables $y = ar/\beta^{1/2}$ leads to

$$\langle f\rangle = \int_0^\infty \frac{y J_0(y\beta^{1/2})\, dy}{p[y^2 + m^2\pi^2 + (pr^2/\alpha_r\beta)]} \qquad \text{(C5)}$$

By analogy to (A17) the inverse Laplace transform of (C5) may be written as

$$f = \int_0^\infty y J_0(y\beta^{1/2}) \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{g(\lambda)}{2\pi i}\, d\lambda\, dy \qquad \text{(C6)}$$

where $g(\lambda) = \{\exp(\lambda t) - 1\}/\{\lambda[y^2 + m^2\pi^2 + (\lambda r^2/\alpha_r\beta)]\}$, γ is a positive real constant, and λ is a complex variable replacing p.

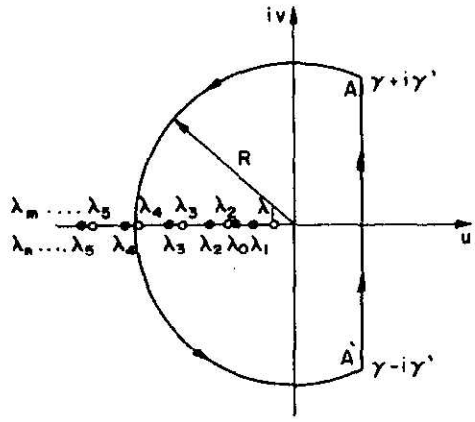

Fig. 7. Contour of integration used with Mellin's inversion formula.

The function $g(\lambda)$ has a removable singularity at $\lambda = 0$ and a simple pole at $\lambda_m = -(y^2 + m^2\pi^2)\alpha_r\beta/r^2$. By using (A18), one finds that the residue of $g(\lambda)$ at $\lambda = \lambda_m$ is

$$\mathrm{Res}|_{\lambda=\lambda_m}$$

$$= [1 - \exp[-t_r\beta(y^2 + m^2\pi^2)]](y^2 + m^2\pi^2)^{-1}$$

and when Cauchy's residue theorem is applied to (C6), the result becomes

$$f = \int_0^\infty y J_0(y\beta^{1/2})\frac{1 - \exp[-t_r\beta(y^2 + m^2\pi^2)]}{y^2 + m^2\pi^2}\, dy \qquad \text{(C7)}$$

By comparing (C4) with (C7) one obtains the general relationship

$$\int_0^\infty y J_0(y\beta^{1/2})\frac{1 - \exp[-t_r\beta(y^2 + m^2\pi^2)]}{y^2 + m^2\pi^2}\, dy$$

$$= \frac{1}{2}\int_{1/4t_r}^\infty \exp\left(-y - \frac{\beta m^2\pi^2}{4y}\right)\frac{dy}{y} \qquad \text{(C8)}$$

$$m = 0, 1, 2, \cdots$$

NOTATION

- a parameter of Hankel transform.
- b initial saturated thickness of aquifer, L.
- d vertical distance between top of perforations in pumping well and initial position of water table, L.
- d_D dimensionless d, equal to d/b.
- I net vertical specific rate of recharge at the water table, LT^{-1}.
- $J_0(x)$ zero-order Bessel function of the first kind.
- K_r horizontal permeability, LT^{-1}.
- K_z vertical permeability, LT^{-1}.
- K_D dimensionless permeability, equal to K_z/K_r.
- l vertical distance between bottom of perforations in pumping well and initial position of water table, L.
- l_D dimensionless l, equal to l/b.
- n_r component of unit outer normal in r direction.
- n_z component of unit outer normal in z direction.
- p parameter of Laplace transform.
- Q pumping rate, L^3T^{-1}.
- r radial distance from pumping well, L.

The inverse Hamkel transform of s^* is obtained directly from (A14) and (B2) in the form

$$\langle s\rangle = \frac{Q}{2\pi T K_D}\int_0^\infty \left(\frac{1}{p} + \frac{\{\sinh[b\eta(1-l_D)] - \sinh[b\eta(1-d_D)]\}\cosh(\eta z)}{(l_D-d_D)\sinh(\eta b)[\alpha_v\eta\sinh(\eta b) + p\cosh(\eta b)]}\right)\frac{aJ_0(ra)}{\eta^2}\,da + \langle\Delta s_0\rangle$$

where the angle brackets denote Laplace transforms. If we define a new variable $y = ar/\beta^{1/2}$, the expression above can be rewritten as

$$\langle s\rangle = \frac{Q}{2\pi T}\int_0^\infty \left(\frac{1}{p} + \frac{\{\sinh[\nu(1-l_D)] - \sinh[\nu(1-d_D)]\}\cosh(\nu z_D)}{(l_D-d_D)\sinh(\nu)[(\alpha_v/b)\nu\sinh(\nu) + p\cosh(\nu)]}\right)\frac{yJ_0(y\beta^{1/2})}{\nu^2}\,dy + \langle\Delta s_0\rangle \tag{A15}$$

where now $\nu^2 = y^2 + (pr^2/\alpha_s\beta)$.

The inversion of the Laplace transform of s is accomplished by using Mellin's inversion formula written as

$$s(r, z, t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}[\exp(\lambda t) - 1]\langle s\rangle(r, z, \lambda)\,d\lambda \tag{A16}$$

where γ is a positive real constant and λ is a complex variable replacing p. If one assumes that the order of integration with respect to λ and y is interchangeable, one obtains from (A15) and (A16)

$$s = \frac{Q}{4\pi T}\int_0^\infty yJ_0(y\beta^{1/2})\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{g(\lambda)}{\pi i}\,d\lambda\,dy + \Delta s_0 \tag{A17}$$

where

$$g(\lambda) = \frac{\exp(\lambda t) - 1}{\xi^2}\left(\frac{1}{\lambda} + \frac{\{\sinh[\xi(1-l_D)] - \sinh[\xi(1-d_D)]\}\cosh(\xi z_D)}{(l_D-d_D)\sinh(\xi)[(\alpha_v/b)\xi\sinh(\xi) + \lambda\cosh(\xi)]}\right)$$

and $\xi^2 = y^2 + (\lambda r^2/\alpha_s\beta)$.

The first step is to determine the singularities of $g(\lambda)$ in the complex domain of λ. Since $g(\lambda)$ has many similarities with the term $f(\lambda)$ in (A10) of *Newman* [1972a, Appendix A], one can immediately deduce that a removable singularity occurs at $\lambda = 0$ and simple poles at $\lambda = -y^2\alpha_s\beta/r^2$, where $\xi = 0$; at $\lambda_0 = (\gamma_0 - y^2)\alpha_s\beta/r^2$, where $\xi = \gamma_0$ and γ_0 is the root of (18); and at an infinite number of locations $\lambda_n = -(\gamma_n^2 + y^2)\alpha_s\beta/r^2$ $(n = 1, 2, \cdots)$, where $\xi^2 = \gamma_n^2$ and γ_n are the roots of (19). In addition, $g(\lambda)$ has infinite number of simple poles at $\lambda_m = -(y^2 + m^2\pi^2)\alpha_s\beta/r^2$ $(m = 1, 2, \cdots)$, where $\xi^2 = -m^2\pi^2$.

The second step is to determine the residues of $g(\lambda)$ at each pole. Using formula

$$\mathrm{Res}|_{\lambda=\lambda_i} = \lim_{\lambda\to\lambda_i}(\lambda - \lambda_i)g(\lambda) \tag{A18}$$

one finds that

$$\mathrm{Res}|_{\lambda=-y^2\alpha_s\beta/r^2} = 0 \tag{A19}$$

$$\mathrm{Res}|_{\lambda=\lambda_0} = 2u_0(y) \tag{A20}$$

$$\mathrm{Res}|_{\lambda=\lambda_n} = 2u_0(y) \tag{A21}$$

where u_0 and u_n have been defined in (17). In addition,

$$\mathrm{Res}|_{\lambda=\lambda_m} = -2\cdot\frac{\{1 - \exp[-t_s\beta(y^2 + m^2\pi^2)]\}}{y^2 + m^2\pi^2}\,\frac{[\sin(m\pi l_D) - \sin(m\pi d_D)]\cos[m\pi(1 - z_D)]}{m\pi(l_D - d_D)} \tag{A22}$$

To evaluate the complex integral in (A17), the path of integration will be as shown in Figure 7. According to Cauchy's residue theorem one has

$$\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{g(\lambda)}{\pi i}\,d\lambda = -\lim_{R\to\infty}\int_A^{A'}\frac{g(\lambda)}{\pi i}\,d\lambda + 2\sum_{n=0}^{\infty}\mathrm{Res}|_{\lambda=\lambda_n} + 2\sum_{m=1}^{\infty}\mathrm{Res}|_{\lambda=\lambda_m} \tag{A23}$$

The integral on the right-hand side of this expression vanishes as $R\to\infty$. Therefore substituting (A20)–(A23) into (A17) and using (C8) to eliminate Δs_0 and the terms arising from the residues at $\lambda = \lambda_m$ lead to the final solution as given by (17).

s drawdown, L.
s_D dimensionless drawdown, equal to $4\pi ts/Q$.
$\langle s \rangle$ Laplace transform of s.
s^* Hankel transform of $\langle s \rangle$.
S storage coefficient, equal to $S_s b$.
S_s specific (elastic) storage, L^{-1}.
S_y specific yield.
t pumping time t.
t_s dimensionless time with respect to S_s, equal to Tt/Sr^2.
t_y dimensionless time with respect to S_y, equal to $Tt/S_y r^2$.
T transmissivity $K_r b$, L^2T^{-1}.
z vertical distance above bottom of aquifer, L.
z_D dimensionless elevation, equal to z/b.
α_s K_r/S_s, L^2T^{-1}.
α_y K_r/S_y, LT^{-1}.
β $K_D r^2/b^2$.
ξ elevation of water table above bottom of aquifer, L.
σ S/S_y.

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