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Growth and Decay of Groundwater-Mounds in Response to Uniform Percolation

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Abstract. The rise and decay of the water table in response to deep percolation depends on the shape of the recharging area, among other factors. Solutions describing this phenomenon when the recharging area is rectangular or circular in shape are developed for the case where the underlying aquifer is effectively infinite in areal extent. The solution when the recharging area is circular is given in terms of a function that can be easily tabulated for a practical range of the parameters. Approximate solutions in terms of already tabulated functions are presented also. The solution when the recharging area is rectangular is obtained in terms of a function that depends on two parameters. This function is tabulated for a wide range of the parameters. This tabulation, as well as approximate relations, affords a means for relatively simple calculation. The solutions are applicable if the rise of the water table relative to the initial depth of saturation does not exceed 50%. (Key words: Groundwater; hydraulics; recharge)

INTRODUCTION

Analytical expressions for the formation of groundwater ridges and mounds beneath spreading basins are available for several cases of this flow phenomenon. Among these are those presented by *Baumann* [1952], *Bittinger and Trelease* [1960], and *Glover* [1961]. Experimental studies have verified the usefulness of these solutions [*Baumann*, 1952; *Marmion*, 1962]. In addition to the usual assumptions with regard to the homogeneity, isotropy, vertical recharge, and the constancy of the formation coefficients, all these solutions assume that the rise of the water table relative to the initial depth of saturation is less than 2%. A solution in terms of the head averaged over the depth of saturation has been obtained by *Hantush* [1963] for the case of an infinitely long recharging area. Experiments using viscous-flow models have shown [*Marino*, 1965, 1967] that this solution is applicable, with relative deviations not exceeding 6%, even when the rise of the water table relative to the initial depth of saturation is as high as 50%.

Artificial recharge by spreading and application of irrigation waters on more or less rectangular or circular areas is not uncommon in practice. The rise and fall of the water table beneath such areas is, therefore, of practical interest. Solutions describing the rise and fall of the water table in response to deep percolation from such areas are developed in the subsequent sections.

ANALYSIS

Rectangular Recharging Area

Figure 1*a* is a diagrammatic representation of a water-table aquifer receiving deep percolation from a rectangular area. The aquifer is effectively infinite in areal extent. Unless otherwise stated, the aquifer is homogeneous, isotropic, and resting on a horizontal impermeable base; the formation coefficients are constant in time and space; and the constant rate of deep percolation relative to the hydraulic conductivity is so small that the vertically downward percolation is almost completely refracted in the direction of the tilt of the water table.

Because of the symmetry of the flow, the main groundwater divides (boundaries across which no flow takes place) lie along the x and y axes.

Statement of the problem. The problem is to determine the rise of the water table in response to a vertically downward uniform rate of percolation that is supplied from a rectangular spreading basin. The water table is assumed to remain below the bottom of the spreading basin.

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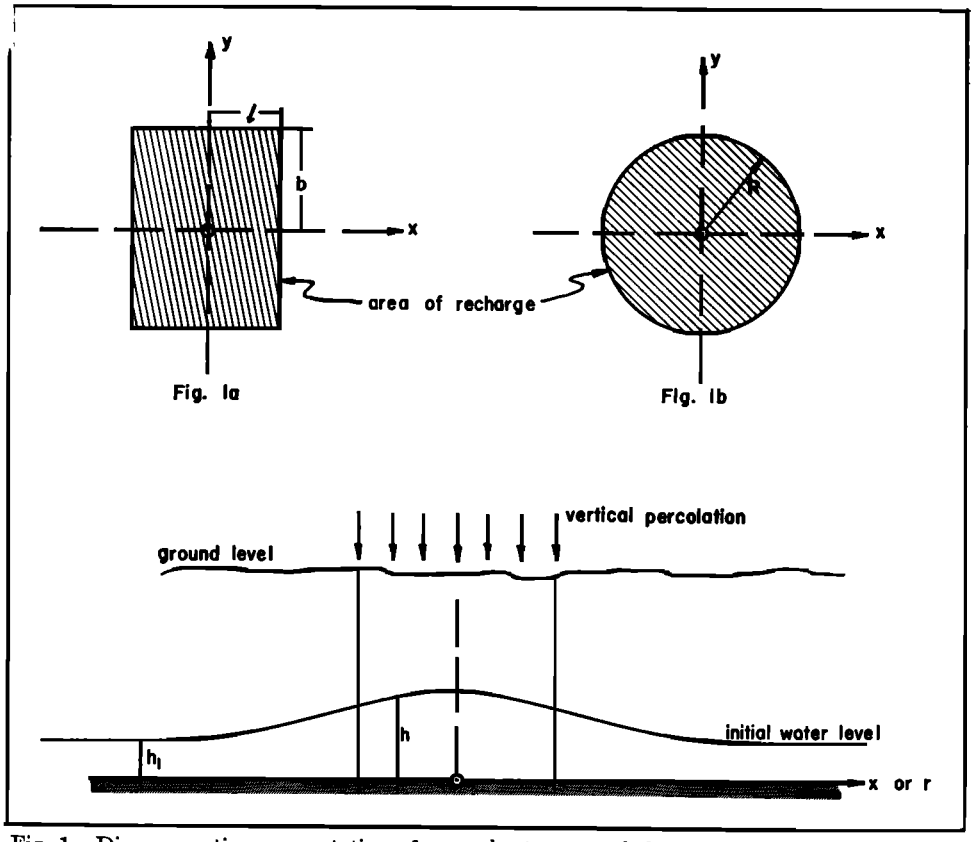

Fig. 1. Diagrammatic representation of groundwater mounds beneath *a*) rectangular recharging area, *b*) circular recharging area.

This flow problem can be approximated by the following [see Figure 1*a* for coordinate system] boundary-value problem:

$$\partial^2 Z/\partial x^2 + \partial^2 Z/\partial y^2 + (2w/K)f(x, y) = (1/\nu)\, \partial Z/\partial t \quad (1)$$

$$Z(x, y, 0) = 0 \quad (2)$$

$$\partial Z(0, y, t)/\partial x = \partial y(x, 0, t)/\partial Z = 0 \quad (3)$$

$$\partial Z(\infty, y, t)/\partial x = \partial Z(x, \infty, t)/\partial y = 0 \quad (4)$$

in which

$$Z = h^2 - h_i^2 \qquad \nu = K\bar{b}/\epsilon \quad (5)$$

$$f(x, y) = f_1(y) \qquad 0 < x < l$$
$$= 0 \qquad x > l \quad (6)$$

with

$$f_1(y) = 1 \qquad 0 < y < a$$
$$= 0 \qquad y > a \quad (7)$$

where $h(x, y, t)$ is the height of the water table above the base of the aquifer, K and ϵ are, respectively, the hydraulic conductivity and the storativity (the specific yield in this instance) of the aquifer; w denotes the constant rate of percolation; $h_i(x, y, \tau_0 + t)$ is the height of the water table that would have prevailed if deep percolation did not occur, assumedly a solution of (1), with h_i replacing Z and $w = 0$; τ_0 is the instant at which percolation occurs; t denotes the time since the incidence of percolation; $2l$ and $2a$ are the dimensions [see Figure 1*a*] of the rectangular strip; and $\bar{b}$ is a constant of linearization [*Hantush*, 1964, equation 93] that can be approximated by

$$\bar{b} = 0.5[h_i(0) + h(t_1)] \quad (8)$$

where t_1 is the period (or subperiod, if it becomes necessary to use the method of successive approximation) at the end of which h is to be estimated.

Solution of the problem. Successive applications of the Laplace transformation with respect to t and the Fourier cosine transformation, with respect to x, then with respect to y (see, for example, *Carslaw and Jaeger* [1959] for the definition and related relations of these transformations), will ultimately yield

$$\bar{Z}_{xy}(\omega, \beta, p) = \frac{2}{\pi} A \frac{\sin \omega l \sin \beta a}{p\omega\beta(p/\nu + \omega^2 + \beta^2]} \quad (9)$$

in which

$$A = 2w/K$$

and $\bar{Z}_{xy}$ is the Fourier cosine transform with respect to y of the Fourier cosine transform with respect to x of the Laplace transform with respect to t of the function $Z(x, y, t)$, and β, ω, and p are, respectively, the parameters of these successive transformations.

Application of the inversion formula for the Fourier cosine transform (with respect to x) on (9) will give

$$\bar{Z}_y(x, \beta, p) = \frac{2}{\pi} A \frac{\sin \beta\omega}{p\beta} \sqrt{2/\pi} \cdot \int_0^\infty \frac{\sin \omega l \cos \omega x}{\omega[\omega^2 + (p/\nu + \beta^2)]} d\omega$$

which, after evaluating the infinite integral [*Erdelyi,* 1954], becomes

$$\bar{Z}_y(x, \beta, p) = \frac{1}{2}\sqrt{\frac{2}{\pi}} A \frac{\sin \beta a}{p\beta(p/\nu + \beta^2)} \cdot \{\exp[-(x - l)\sqrt{p/\nu + \beta^2}] - \exp[-(l + x)\sqrt{p/\nu + \beta^2}]\} \quad \text{if} \quad l < x \quad (10)$$

or

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \frac{A \sin \beta a}{p\beta(p/\nu + \beta^2)} \cdot \{2 - \exp[-(l - x)\sqrt{p/\nu + \beta^2}] - \exp[-(l + x)\sqrt{p/\nu + \beta^2}]\} \quad \text{if} \quad l > x \quad (11)$$

The inverse Laplace transform of (10) or (11) can each be shown, using the translation theorem, the convolution theorem, and a table of inverse transforms, to be given by

$$Z_y(x, \beta, t) = \frac{1}{2}\sqrt{\frac{2}{\pi}} \nu A \frac{\sin \beta a}{\beta} \int_0^t \left\{\operatorname{erf}\left(\frac{l + x}{\sqrt{4\nu\tau}}\right) + \operatorname{erf}\left(\frac{l - x}{\sqrt{4\nu\tau}}\right)\right\} \exp(-\nu\tau\beta^2)\, d\tau \quad (12)$$

in which $\operatorname{erf}(x) = 1 - \operatorname{erfc}(x)$, erf and erfc being the error function and its complement, respectively.

Applying the inversion formula of the Fourier cosine transform (with respect to y) on (12) and interchanging the order of integrations will yield

$$Z(x, y, t) = \frac{\nu A}{2} \int_0^t \left\{\left[\operatorname{erf}\left(\frac{l + x}{\sqrt{4\nu\tau}}\right) + \operatorname{erf}\left(\frac{l - x}{\sqrt{4\nu\tau}}\right)\right] \cdot \frac{2}{\pi} \int_0^\infty \exp(-\nu\tau\beta^2) \frac{\sin \beta a}{\beta} \cos \beta y\, d\beta\right\} d\tau$$

which, after evaluating the infinite integral [*Erdelyi,* 1954], can be written as

$$Z(x, y, t) = \frac{\nu A}{2} \int_0^t \left[\operatorname{erf}\left(\frac{l + x}{\sqrt{4\nu\tau}}\right) + \operatorname{erf}\left(\frac{l - x}{\sqrt{4\nu\tau}}\right)\right] \cdot \left[\operatorname{erf}\left(\frac{a + y}{\sqrt{4\nu\tau}}\right) + \operatorname{erf}\left(\frac{a - y}{\sqrt{4\nu\tau}}\right)\right] d\tau \quad (13)$$

Water-table rise. After changing the variable of integration from τ to $t\tau$ in (13), the water-table rise can be finally expressed as

$$h^2 - h_i^2 = (w/2K)(\nu t) \cdot \left\{S^*\left(\frac{l + x}{\sqrt{4\nu t}}, \frac{a + y}{\sqrt{4\nu t}}\right) + S^*\left(\frac{l + x}{\sqrt{4\nu t}}, \frac{a - y}{\sqrt{4\nu t}}\right) + S^*\left(\frac{l - x}{\sqrt{4\nu t}}, \frac{a + y}{\sqrt{4\nu t}}\right) + S^*\left(\frac{l - x}{\sqrt{4\nu t}}, \frac{a - y}{\sqrt{4\nu t}}\right)\right\} \quad (14)$$

where the function S^* is defined by

$$S^*(\alpha, \beta) = \int_0^1 \operatorname{erf}\left(\frac{\alpha}{\sqrt{\tau}}\right) \cdot \operatorname{erf}\left(\frac{\beta}{\sqrt{\tau}}\right) d\tau \quad (15)$$

a function that can be integrated by parts to

Values of the function $S^*(\alpha,\beta) = \int_0^1 \mathrm{erf}\left(\frac{\alpha}{\sqrt{\tau}}\right) \mathrm{erf}\left(\frac{\beta}{\sqrt{\tau}}\right) d\tau$

α \ β	0.02	0.04	0.06	0.08	0.10	0.14	0.18	0.22	0.26	0.30	0.34	0.38	0.42	0.46	0.50	0.54	0.58	0.62
0.02	0.0041	0.0073	0.0101	0.0125	0.0146	0.0184	0.0216	0.0243	0.0267	0.0288	0.0306	0.0322	0.0337	0.0349	0.0361	0.0371	0.0380	0.0387
0.04	0.0073	0.0135	0.0188	0.0236	0.0278	0.0353	0.0416	0.0470	0.0518	0.0559	0.0596	0.0628	0.0657	0.0683	0.0705	0.0725	0.0743	0.0759
0.06	0.0101	0.0188	0.0266	0.0335	0.0398	0.0509	0.0602	0.0684	0.0754	0.0817	0.0871	0.0920	0.0963	0.1001	0.1035	0.1065	0.1091	0.1115
0.08	0.0125	0.0236	0.0335	0.0425	0.0508	0.0652	0.0776	0.0884	0.0978	0.1060	0.1133	0.1197	0.1254	0.1305	0.1350	0.1389	0.1425	0.1456
0.10	0.0146	0.0278	0.0398	0.0508	0.0608	0.0786	0.0939	0.1072	0.1188	0.1290	0.1381	0.1461	0.1532	0.1595	0.1650	0.1700	0.1744	0.1783
0.14	0.0184	0.0353	0.0509	0.0652	0.0786	0.1025	0.1232	0.1414	0.1573	0.1714	0.1839	0.1949	0.2048	0.2135	0.2212	0.2281	0.2343	0.2397
0.18	0.0216	0.0416	0.0602	0.0776	0.0939	0.1232	0.1490	0.1716	0.1916	0.2094	0.2251	0.2391	0.2515	0.2626	0.2724	0.2812	0.2890	0.2959
0.22	0.0243	0.0470	0.0684	0.0884	0.1072	0.1414	0.1716	0.1984	0.2222	0.2433	0.2621	0.2789	0.2938	0.3071	0.3189	0.3295	0.3389	0.3472
0.26	0.0267	0.0518	0.0754	0.0978	0.1188	0.1573	0.1916	0.2222	0.2494	0.2737	0.2954	0.3147	0.3320	0.3474	0.3612	0.3735	0.3844	0.3941
0.30	0.0288	0.0559	0.0817	0.1060	0.1290	0.1714	0.2094	0.2433	0.2737	0.3009	0.3252	0.3470	0.3665	0.3839	0.3995	0.4134	0.4257	0.4368
0.34	0.0306	0.0596	0.0871	0.1133	0.1381	0.1839	0.2251	0.2621	0.2954	0.3252	0.3520	0.3761	0.3976	0.4169	0.4341	0.4495	0.4633	0.4756
0.38	0.0322	0.0628	0.0920	0.1197	0.1461	0.1949	0.2391	0.2789	0.3147	0.3470	0.3761	0.4022	0.4256	0.4466	0.4654	0.4823	0.4973	0.5108
0.42	0.0337	0.0657	0.0963	0.1254	0.1532	0.2048	0.2515	0.2938	0.3320	0.3665	0.3976	0.4256	0.4508	0.4734	0.4937	0.5119	0.5281	0.5427
0.46	0.0349	0.0683	0.1001	0.1305	0.1595	0.2135	0.2626	0.3071	0.3474	0.3839	0.4169	0.4466	0.4734	0.4975	0.5191	0.5385	0.5559	0.5715
0.50	0.0361	0.0705	0.1035	0.1350	0.1650	0.2212	0.2724	0.3189	0.3612	0.3995	0.4341	0.4654	0.4937	0.5191	0.5420	0.5626	0.5810	0.5975
0.54	0.0371	0.0725	0.1065	0.1389	0.1700	0.2281	0.2812	0.3295	0.3735	0.4134	0.4495	0.4823	0.5119	0.5385	0.5626	0.5842	0.6036	0.6209
0.58	0.0380	0.0743	0.1091	0.1425	0.1744	0.2343	0.2890	0.3389	0.3844	0.4257	0.4633	0.4973	0.5281	0.5559	0.5810	0.6036	0.6238	0.6420
0.62	0.0387	0.0759	0.1115	0.1456	0.1783	0.2397	0.2959	0.3472	0.3941	0.4368	0.4756	0.5108	0.5427	0.5715	0.5975	0.6209	0.6420	0.6609
0.66	0.0394	0.0773	0.1136	0.1484	0.1818	0.2445	0.3020	0.3547	0.4027	0.4466	0.4865	0.5227	0.5556	0.5854	0.6122	0.6364	0.6582	0.6778
0.70	0.0401	0.0785	0.1154	0.1509	0.1849	0.2488	0.3075	0.3612	0.4104	0.4553	0.4962	0.5334	0.5672	0.5977	0.6254	0.6503	0.6728	0.6929
0.74	0.0406	0.0796	0.1171	0.1531	0.1876	0.2526	0.3123	0.3671	0.4172	0.4630	0.5048	0.5429	0.5774	0.6087	0.6371	0.6627	0.6857	0.7064
0.78	0.0411	0.0806	0.1185	0.1550	0.1900	0.2559	0.3166	0.3722	0.4232	0.4699	0.5125	0.5513	0.5865	0.6185	0.6475	0.6736	0.6972	0.7184
0.82	0.0415	0.0814	0.1198	0.1567	0.1921	0.2589	0.3203	0.3768	0.4286	0.4760	0.5192	0.5587	0.5946	0.6272	0.6567	0.6834	0.7074	0.7291
0.86	0.0419	0.0822	0.1209	0.1582	0.1940	0.2615	0.3237	0.3808	0.4333	0.4813	0.5252	0.5653	0.6017	0.6348	0.6648	0.6920	0.7165	0.7386
0.90	0.0422	0.0828	0.1219	0.1595	0.1957	0.2638	0.3266	0.3844	0.4374	0.4860	0.5305	0.5711	0.6080	0.6416	0.6721	0.6996	0.7245	0.7469
0.94	0.0425	0.0834	0.1228	0.1607	0.1971	0.2658	0.3292	0.3875	0.4411	0.4902	0.5351	0.5762	0.6136	0.6476	0.6784	0.7063	0.7316	0.7543
0.98	0.0428	0.0839	0.1236	0.1617	0.1984	0.2676	0.3314	0.3902	0.4442	0.4938	0.5392	0.5807	0.6184	0.6528	0.6840	0.7123	0.7378	0.7608
1.00	0.0429	0.0842	0.1239	0.1622	0.1990	0.2684	0.3324	0.3914	0.4457	0.4955	0.5410	0.5827	0.6206	0.6552	0.6865	0.7150	0.7406	0.7638
1.20	0.0437	0.0858	0.1263	0.1654	0.2030	0.2740	0.3396	0.4001	0.4558	0.5070	0.5540	0.5969	0.6362	0.6719	0.7044	0.7339	0.7605	0.7846
1.40	0.0441	0.0866	0.1275	0.1669	0.2049	0.2767	0.3431	0.4043	0.4608	0.5127	0.5603	0.6039	0.6438	0.6801	0.7132	0.7432	0.7704	0.7949
1.80	0.0444	0.0871	0.1283	0.1680	0.2062	0.2785	0.3454	0.4071	0.4641	0.5165	0.5645	0.6086	0.6489	0.6856	0.7190	0.7494	0.7769	0.8018
2.00	0.0444	0.0871	0.1284	0.1681	0.2064	0.2787	0.3457	0.4075	0.4645	0.5169	0.5651	0.6092	0.6495	0.6863	0.7198	0.7502	0.7778	0.8027
2.20	0.0444	0.0872	0.1284	0.1682	0.2065	0.2788	0.3458	0.4076	0.4646	0.5171	0.5653	0.6094	0.6497	0.6865	0.7200	0.7505	0.7781	0.8030
2.50	0.0444	0.0872	0.1284	0.1682	0.2065	0.2788	0.3458	0.4077	0.4647	0.5172	0.5653	0.6095	0.6498	0.6867	0.7202	0.7506	0.7782	0.8032
3.00	0.0444	0.0872	0.1284	0.1682	0.2065	0.2789	0.3458	0.4077	0.4647	0.5172	0.5654	0.6095	0.6499	0.6867	0.7202	0.7506	0.7782	0.8032

α \ β	0.62	0.66	0.70	0.74	0.78	0.82	0.86	0.90	0.94	0.98	1.00	1.20	1.40	1.80	2.00	2.20	2.50	3.00
0.02	0.0387	0.0394	0.0401	0.0406	0.0411	0.0415	0.0419	0.0422	0.0425	0.0428	0.0429	0.0437	0.0441	0.0444	0.0444	0.0444	0.0444	0.0444
0.04	0.0759	0.0773	0.0785	0.0796	0.0806	0.0814	0.0822	0.0828	0.0834	0.0839	0.0842	0.0858	0.0866	0.0871	0.0871	0.0872	0.0872	0.0872
0.06	0.1115	0.1136	0.1154	0.1171	0.1185	0.1198	0.1209	0.1219	0.1228	0.1236	0.1239	0.1263	0.1275	0.1283	0.1284	0.1284	0.1284	0.1284
0.08	0.1456	0.1484	0.1509	0.1531	0.1550	0.1567	0.1582	0.1595	0.1607	0.1617	0.1622	0.1654	0.1669	0.1680	0.1681	0.1682	0.1682	0.1682
0.10	0.1783	0.1818	0.1849	0.1876	0.1900	0.1921	0.1940	0.1957	0.1971	0.1984	0.1990	0.2030	0.2049	0.2062	0.2064	0.2065	0.2065	0.2065
0.14	0.2397	0.2445	0.2488	0.2526	0.2559	0.2589	0.2615	0.2638	0.2658	0.2676	0.2684	0.2740	0.2767	0.2785	0.2787	0.2788	0.2788	0.2789
0.18	0.2959	0.3020	0.3075	0.3123	0.3166	0.3203	0.3237	0.3266	0.3292	0.3314	0.3324	0.3396	0.3431	0.3454	0.3457	0.3458	0.3458	0.3458
0.22	0.3472	0.3547	0.3612	0.3671	0.3722	0.3768	0.3808	0.3844	0.3875	0.3902	0.3914	0.4001	0.4043	0.4071	0.4075	0.4076	0.4077	0.4077
0.26	0.3941	0.4027	0.4104	0.4172	0.4232	0.4286	0.4333	0.4374	0.4411	0.4442	0.4457	0.4558	0.4608	0.4641	0.4645	0.4646	0.4647	0.4647
0.30	0.4368	0.4466	0.4553	0.4630	0.4699	0.4760	0.4813	0.4860	0.4902	0.4938	0.4955	0.5070	0.5127	0.5165	0.5169	0.5171	0.5172	0.5172
0.34	0.4756	0.4865	0.4962	0.5048	0.5125	0.5192	0.5252	0.5305	0.5351	0.5392	0.5410	0.5540	0.5603	0.5645	0.5651	0.5653	0.5653	0.5654
0.38	0.5108	0.5227	0.5334	0.5429	0.5513	0.5587	0.5653	0.5711	0.5762	0.5807	0.5827	0.5969	0.6039	0.6086	0.6092	0.6094	0.6095	0.6095
0.42	0.5427	0.5556	0.5672	0.5774	0.5865	0.5946	0.6017	0.6080	0.6136	0.6184	0.6206	0.6362	0.6438	0.6489	0.6495	0.6497	0.6498	0.6499
0.46	0.5715	0.5854	0.5977	0.6087	0.6185	0.6272	0.6348	0.6416	0.6476	0.6528	0.6552	0.6719	0.6801	0.6856	0.6863	0.6865	0.6867	0.6867
0.50	0.5975	0.6122	0.6254	0.6371	0.6475	0.6567	0.6648	0.6721	0.6784	0.6840	0.6865	0.7044	0.7132	0.7190	0.7198	0.7200	0.7202	0.7202
0.54	0.6209	0.6364	0.6503	0.6627	0.6736	0.6834	0.6920	0.6996	0.7063	0.7123	0.7150	0.7339	0.7432	0.7494	0.7502	0.7505	0.7506	0.7506
0.58	0.6420	0.6582	0.6728	0.6857	0.6972	0.7074	0.7165	0.7245	0.7316	0.7378	0.7406	0.7605	0.7704	0.7769	0.7778	0.7781	0.7782	0.7782
0.62	0.6609	0.6778	0.6929	0.7064	0.7184	0.7291	0.7386	0.7469	0.7543	0.7608	0.7638	0.7846	0.7949	0.8018	0.8027	0.8030	0.8032	0.8032
0.66	0.6778	0.6953	0.7110	0.7250	0.7375	0.7486	0.7584	0.7671	0.7748	0.7816	0.7846	0.8064	0.8171	0.8243	0.8252	0.8255	0.8257	0.8257
0.70	0.6929	0.7110	0.7272	0.7417	0.7546	0.7660	0.7762	0.7852	0.7932	0.8002	0.8034	0.8259	0.8370	0.8445	0.8454	0.8458	0.8460	0.8460
0.74	0.7064	0.7250	0.7417	0.7566	0.7698	0.7816	0.7921	0.8014	0.8096	0.8168	0.8201	0.8434	0.8549	0.8627	0.8636	0.8640	0.8642	0.8642
0.78	0.7184	0.7375	0.7546	0.7698	0.7834	0.7956	0.8063	0.8159	0.8243	0.8317	0.8351	0.8591	0.8710	0.8789	0.8799	0.8803	0.8805	0.8805
0.82	0.7291	0.7486	0.7660	0.7816	0.7956	0.8080	0.8190	0.8288	0.8374	0.8450	0.8485	0.8731	0.8853	0.8935	0.8945	0.8949	0.8951	0.8951
0.86	0.7386	0.7584	0.7762	0.7921	0.8063	0.8190	0.8302	0.8402	0.8491	0.8569	0.8604	0.8855	0.8980	0.9065	0.9075	0.9079	0.9081	0.9081
0.90	0.7469	0.7671	0.7852	0.8014	0.8159	0.8288	0.8402	0.8504	0.8594	0.8674	0.8710	0.8966	0.9094	0.9180	0.9191	0.9195	0.9197	0.9197
0.94	0.7543	0.7748	0.7932	0.8096	0.8243	0.8374	0.8491	0.8594	0.8686	0.8767	0.8803	0.9064	0.9195	0.9282	0.9294	0.9298	0.9300	0.9300
0.98	0.7608	0.7816	0.8002	0.8168	0.8317	0.8450	0.8569	0.8674	0.8767	0.8849	0.8886	0.9151	0.9284	0.9373	0.9384	0.9389	0.9391	0.9391
1.00	0.7638	0.7846	0.8034	0.8201	0.8351	0.8485	0.8604	0.8710	0.8803	0.8886	0.8924	0.9191	0.9324	0.9414	0.9426	0.9430	0.9432	0.9433
1.20	0.7846	0.8064	0.8259	0.8434	0.8591	0.8731	0.8855	0.8966	0.9064	0.9151	0.9191	0.9472	0.9614	0.9709	0.9722	0.9726	0.9728	0.9729
1.40	0.7949	0.8171	0.8370	0.8549	0.8710	0.8853	0.8980	0.9094	0.9195	0.9284	0.9324	0.9614	0.9759	0.9858	0.9871	0.9875	0.9878	0.9878
1.80	0.8018	0.8243	0.8445	0.8627	0.8789	0.8935	0.9065	0.9180	0.9282	0.9373	0.9414	0.9709	0.9858	0.9959	0.9972	0.9977	0.9979	0.9980
2.00	0.8027	0.8252	0.8454	0.8636	0.8799	0.8945	0.9075	0.9191	0.9294	0.9384	0.9426	0.9722	0.9871	0.9972	0.9985	0.9990	0.9992	0.9993
2.20	0.8030	0.8255	0.8458	0.8640	0.8803	0.8949	0.9079	0.9195	0.9298	0.9389	0.9430	0.9726	0.9875	0.9977	0.9990	0.9995	0.9997	0.9998
2.50	0.8032	0.8257	0.8460	0.8642	0.8805	0.8951	0.9081	0.9197	0.9300	0.9391	0.9432	0.9728	0.9878	0.9979	0.9992	0.9997	1.0000	1.0000
3.00	0.8032	0.8257	0.8460	0.8642	0.8805	0.8951	0.9081	0.9197	0.9300	0.9391	0.9433	0.9729	0.9878	0.9980	0.9993	0.9998	1.0000	1.0000

yield an expression in terms of tabulated functions [see (23)]. However, computation in the resulting expression is cumbersome. To obviate this cumbersomeness, Table 1 is provided, with a wide range of α and β. For values of α and β beyond the range of Table 1, the function can be evaluated by the relations subsequent to (25).

Water-table decay. If percolation ceases at the instant t_0 since incidence, the water-table decay can be obtained by superposing hypothetically on the flow system at $t = t_0$ a rate of uniform discharge equal to that of the uniform rate of percolation. This is possible, since the differential equation is linear in h^2. Thus, the decay of the water table is given by

$$h^2 - h_i^2 = Z(x, y, t) - Z(x, y, t - t_0) \quad (16)$$

where $Z(x, y, t)$ is the right-hand member of (14).

Maximum height of water-table. The maximum height of the water table in the flow system of Figure 1 occurs underneath the center of the recharging area. Thus, from (14) and (16), with $x = y = 0$, the maximum height of the water table during its rise and decay is given, respectively, by

$$h_m^2 - h_i^2 = (2w/K)\nu t S^*(l/\sqrt{4\nu t} \quad a/\sqrt{4\nu t}) \quad (17)$$

and

$$h_m^2 - h_i^2 = (2w/K)\left\{\nu t S^*\left(\frac{l}{\sqrt{4\nu t}} \quad \frac{a}{\sqrt{4\nu t}}\right) - \nu t' S^*\left(\frac{l}{\sqrt{4\nu t'}} \quad \frac{a}{\sqrt{4\nu t'}}\right)\right\} \quad (18)$$

in which $t' = t - t_0$ and h_m is the maximum height of the water table.

Solution for an infinitely long recharging strip. Fluctuation of the water table beneath an infinitely long recharging strip is a special case of (14). When a becomes infinitely large, the flow is effectively in the x direction. Since

$$S^*(\alpha, \infty) = \int_0^1 \operatorname{erf}(\alpha/\sqrt{\tau})\, d\tau = 1 - 4\mathrm{i}^2 \operatorname{erfc}(\alpha) \quad (19)$$

and

$$S^*(-\alpha, \infty) = -\int_0^1 \operatorname{erf}\left(\frac{\alpha}{\sqrt{\tau}}\right) d\tau = -1 + 4\mathrm{i}^2 \operatorname{erfc}(\alpha) \quad (20)$$

where i^2 erfc is the second repeated integral of the error function [see *Carslaw and Jaeger*, 1959, for tables], equation 14 with $a \to \infty$, for $x < l$, will become

$$h^2 - h_i^2 = (w/K)\nu t\left\{2 - 4\mathrm{i}^2 \operatorname{erfc}\left(\frac{l - x}{\sqrt{4\nu t}}\right) - 4\mathrm{i}^2 \operatorname{erfc}\left(\frac{l + x}{\sqrt{4\nu t}}\right)\right\} \quad (21)$$

and, for $x > l$

$$h^2 - h_i^2 = (w/K)\nu t\left\{4\mathrm{i}^2 \operatorname{erfc}\left(\frac{x - l}{\sqrt{4\nu t}}\right) - 4\mathrm{i}^2 \operatorname{erfc}\left(\frac{l + x}{\sqrt{4\nu t}}\right)\right\} \quad (22)$$

a result that has already been obtained [*Hantush*, 1963; *Marino*, 1967].

The function $S^*(\alpha, \beta)$. Integrating by parts, (15) can be put as

$$S^*(\alpha, \beta) = \operatorname{erf}(\alpha) \operatorname{erf}(\beta) + (4/\pi)\alpha\beta W(\alpha^2 + \beta^2) + (2/\sqrt{\pi})\,[\alpha e^{-\alpha^2} \operatorname{erf}(\beta) + \beta e^{-\beta^2} \operatorname{erf}(\alpha)] - 2[\alpha^2 M^*(\beta/\alpha, \alpha^2) + \beta^2 M^*(\alpha/\beta, \beta^2) \quad (23)$$

where $W(x)$ is the well function for nonleaky aquifers, known in the mathematical literature as the negative exponential integral of $(-x)$, and $M^*(\alpha, \beta)$ is a function defined by

$$M^*(\alpha, \beta) = \frac{2}{\pi}\int_0^\alpha \exp\left[-\beta(1 + y^2)\right. \frac{dy}{1 + y^2} \quad (24)$$

and is available in tabular form [*Hantush*, 1967].

For values of α and β beyond the range of Table 1, the following relations and approximations for $S^*(\alpha, \beta)$, in addition to (23), are useful for practical computations:

$$S^*(-\alpha, \beta) = S^*(\alpha, -\beta) = -S^*(\alpha, \beta)$$

$$S^*(-\alpha, -\beta) = S^*(\alpha, \beta); \; S^*(\alpha, \beta) = S^*(\beta, \alpha)$$

$$S^*(0, \beta) = S^*(\alpha, 0) = 0$$

$$S^*(\alpha, \beta) \simeq 1 - 4i^2 \operatorname{erfc}(\beta), \quad \text{if} \quad \alpha \geq 3$$

$$S^*(\alpha, \beta) \simeq 1 - 4i^2 \operatorname{erfc}(\alpha), \quad \text{if} \quad \beta \geq 3$$

$$S^*(\alpha, \beta) \simeq 1, \quad \text{if} \quad \alpha \geq 3 \quad \text{and} \quad \beta \geq 3$$

$$S^*(\alpha, \beta) \simeq \frac{4}{\pi}\alpha\beta\left\{3 + W(\alpha^2 + \beta^2) - \left[\frac{\alpha}{\beta}\tan^{-1}\frac{\beta}{\alpha} + \frac{\beta}{\alpha}\tan^{-1}\frac{\alpha}{\beta}\right]\right\}$$

$$\text{if} \quad \alpha^2 + \beta^2 \leq 0.10 \qquad (25)$$

Circular Recharging Area

The flow underneath a circular uniformly recharging area can be approximated [see Figure 1*b*] by the following boundary-value problem:

$$\frac{\partial^2 Z}{\partial r^2} + \frac{1}{r}\frac{\partial Z}{\partial r} + \frac{2W}{K}f(r) = \frac{1}{r}\frac{\partial Z}{\partial t} \qquad (26)$$

$$Z(r, 0) = 0$$

$$\partial Z(0, t)/\partial r = 0$$

$$s(\infty, t) = 0$$

in which

$$Z = h^2 - h_i^2 \qquad \nu = K\bar{b}/\epsilon$$

and

$$f(r) = 1 \qquad 0 < r < R$$

$$= 0 \qquad r > R$$

where R denotes the radius of the circular recharging area, r is the radial distance to any point measured from the center of the recharging area, and other symbols have been defined.

Water-table rise. The preceding boundary-value problem is solved by using the Laplace and the zero-ordered Hankel transformations [see for example, *Carslaw and Jaeger*, 1959]. The final result can be put as

$$h^2 - h_0^2 = \frac{2V}{\pi K}\int_0^\infty (1 - e^{-q\beta^2})J_1(\beta)J_0(\rho\beta)\frac{d\beta}{\beta^2}$$

$$= (2V/\pi K)f(q, \rho) \qquad (27)$$

in which

$$V = w\pi R^2 \qquad q = \nu t/R^2 \qquad \rho = r/R$$

and J_0, J_1 are, respectively, the zero and first ordered Bessel functions of the first kind.

The function $f(q, \rho)$ is not available in tabular form. It can, however, be easily tabulated for a practical range of q and ρ.

Maximum height of water-table. The maximum height of the water-table takes place beneath the center of the recharging area. From (27), with $r = 0$, the solution (using Laplace transformation in evaluating infinite integrals) can finally be reduced to

$$h_m^2 - h_i^2 = \frac{V}{2\pi K}[W(u_0) + (1 - e^{-u_0})/u_0] \qquad (28)$$

which, if $u_0 \leq 0.05$, can be written as

$$h_m^2 - h_i^2 = (V/2\pi K)[W(u_0) + 1]$$

$$= (V/2\pi K)\ln(6.11\nu t/R^2) \qquad (29)$$

where

$$u_0 = R^2/4\nu t \qquad (30)$$

and $W(u)$ is the well function for nonleaky aquifers.

Approximate water-table rise. When $t \geq 0.5\, r^2/\nu$ (that is, $u \leq 0.5$), the water-table rise (using Laplace transformation in evaluating integrals) can be approximated for $r < R$ by

$$h^2 - h_i^2 = \frac{V}{2\pi K}\left\{W(u_0) - \left(\frac{r}{R}\right)^2 e^{-u_0} + \frac{1}{u_0}(1 - e^{-u_0})\right\} \qquad (31)$$

and, for $t \geq 0.5\, R^2/\nu$ and $r > R$, by

$$h^2 - h_i^2 = (V/2\pi K)\{W(u) + 0.5u_0e^{-u}\} \qquad (32)$$

where

$$u = r^2/4\nu t$$

and other symbols have been defined.

Water-table decay. The principle used in obtaining (16) can be used to obtain the water-table decay beneath a circular recharging area. This decay is given by

$$h^2 - h_i^2 = Z(r, t) - Z(r, t - t_0) \qquad (33)$$

in which $Z(r, t)$ is the right-hand member of the pertinent expression for the water-table rise, and t_0 is the period of continuous recharge. When using (31) and (32) in the preceding equation, the time criteria become, respectively, $(t - t_0) > 0.5\, r^2/\nu$ and $(t - t_0) > 0.5\, R^2/\nu$.

CONCLUDING REMARKS

The solutions presented previously provide a means, not only for forecasting the rise and decay of the water table, but also for determining, through analyses of observed water levels, two of the three flow parameters if one of these parameters is known. The three parameters are the rate of uniform recharge, the storativity, and the hydraulic conductivity. The classical procedure of type-curve methods [see, for example, *Todd,* 1959; *Hantush,* 1964; *De Wiest,* 1965; *Davis and De Wiest,* 1966] that are used in analyzing data from pumping tests can be adapted to analyze observed profiles and fluctuations of water levels beneath recharging areas when suitable expressions for such profiles and fluctuations are available. For the cases treated in the present study, the pertinent expressions are those given by (14), (21), (22), (27), (31), and (32).

When the recharging area is near other boundaries such as impermeable boundaries and/or lines of constant head, the method of images [see, for example, *Todd,* 1959; *Hantush,* 1964] can be applied to obtain the required solutions near such boundaries, by using the solutions for aquifers of infinite areal extent.

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