## APPENDIX.—NOTATION

The following symbols, adopted for use in this paper, conform essentially with "American Standard Letter Symbols for Hydraulics" (ASA Z10.2-1942), prepared by a committee of the American Standards Association with Society representation, and approved by the Association in 1942:

$A$ = the cross-sectional area of flow, in square feet;

$C_1$ = a coefficient defined by

$$\left(\frac{8\ g\ \phi_o}{f}\right)^{1/2}$$ ; in square root of feet per second;

$C_o$, $C'_o$ = experimental coefficients;

$f$ = the Darcy friction factor;

$g$ = the acceleration of gravity, in feet per second per second;

$N$ = a ratio defined by measured flow rate to theoretical flow rate;

$\overline{N}$ = an average of several ratios defined above;

$n$ = the coefficient in Manning's formula;

$Q$ = the rate of flow, in cubic feet per second (cfs);

$R$ = the hydraulic radius, in feet;

$\mathbf{R}$ = the Reynolds number;

$u$ = the x-component of velocity (local), in feet per second;

$\bar{u}$ = the x-component of velocity (average in vertical), in feet per second;

$\bar{u}_{exp.max.}$ = the x-component of velocity (maximum experimental—average in vertical), in feet per second;

$\bar{\bar{u}}$ = the x-component of velocity (average of section), in feet per second;

$x,y,z$ = the coordinate axes and distances measured parallel thereto;

$y_o, z_o$ = maximum distances, in feet;

$\theta_o$ = the transverse slope;

$\nu$ = the kinematic viscosity of water, in square feet per second; and

$\phi_o$ = the longitudinal slope, in percent.

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# AQUIFER TESTS ON PARTIALLY PENETRATING WELLS

By Mahdi S. Hantush[1]

## SYNOPSIS

The theory of nonsteady flow toward artesian wells partially penetrating infinite nonleaky aquifers is used to outline methods for the determination of the formation coefficients as well as the thickness of the water-bearing formation. Applications of these methods are illustrated by analyzing data from ground-water basins in New Mexico.

## INTRODUCTION

In analyzing nonsteady flow problems of partial penetration, the practice is to make use of complete penetration formulas with or without adjustments, depending on the nature of the flow system considered.[2] These adjustments, if used, are based on the steady-state solutions of partial penetration. Such procedures may give fair results if the aquifer penetrated is of small and known thickness, provided that the assumptions made in each case are obtained. However, in the case of thick aquifers or aquifers of unknown thicknesses, such procedures are inapplicable.

Methods for determining the hydraulic properties, as well as the thickness, of a water-bearing formation by using aquifer test data obtained from wells

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2 "Engineering Hydrology," by S. S. Butler, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1957.



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that partially penetrate the formation are outlined herein. The procedures are based on the theory developed by the writer[3] for the nonsteady flow toward a steadily discharging well that penetrates an artesian aquifer of infinite areal extent.

*Notation.*—The letter symbols adopted for use in this paper are defined where they first appear, in the illustrations or in the text, and are arranged alphabetically, for convenience of reference, in the Appendix.

## DRAWDOWN EQUATIONS

The equations that provide the drawdown distribution in and near partially penetrating artesian wells and that are useful in analyzing data from aquifer tests, will be presented. Details of other equations are available.[3]

*Drawdown in Piezometers.*—Piezometers are small-diameter pipes driven into an aquifer, so that entrance of water into the pipes is solely from the bottom. The drawdown s in a piezometer having a depth of penetration z and being a distance r from the center of a pumped well that is screen between the depths d and l, and whose discharge Q is constant, is, for the case of a homogeneous, isotropic, and nonleaky elastic aquifer, infinite in areal extent and of uniform thickness b, as given below, and is valid only during the period of time specified. (See Fig. 1 for the coordinate system.)

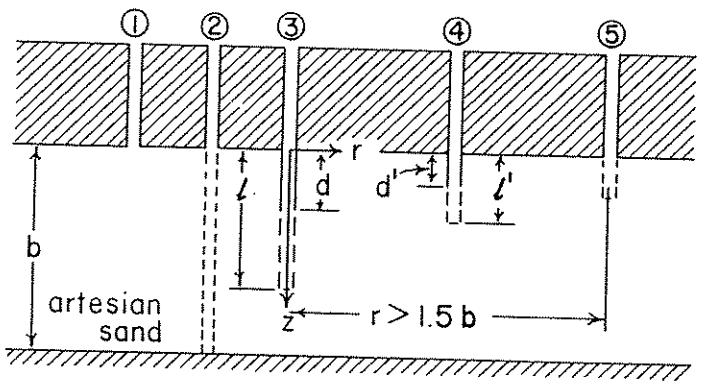


FIG. 1.—DIAGRAMMATIC REPRESENTATION OF PARTIALLY PENETRATING WELLS IN ARTESIAN AQUIFER

For a relatively short period of pumping, the equation of drawdown for $t < \frac{(2b - l - z)^2 S_s}{20K}$ is

$$s = \frac{Q}{8 \pi K (l - d)} E \left(u, \frac{l}{r}, \frac{d}{r}, \frac{z}{r}\right) \quad \ldots\ldots\ldots\ldots (1)$$

[3] "Drawdown Around a Partially Penetrating Well," by M. S. Hantush, Proceedings, ASCE, Vol. 87, No. HY 4, July, 1961.



in which

$$E = M\left[\frac{u, (l + z)}{r}\right] - M\left[\frac{u, (d + z)}{r}\right] + M\left[\frac{u, (l - z)}{r}\right] - M\left[\frac{u, (d - z)}{r}\right] \quad \ldots\ldots\ldots\ldots (2)$$

and

$$u = \left(\frac{r^2 S_s}{4 K t}\right) \quad \ldots\ldots\ldots\ldots (3)$$

t is the time since pumping started, $S_s$ and K are the specific storage (volume of water released from storage in a unit volume of the aquifer under a unit head decline, of dimension $L^{-1}$) and the hydraulic conductivity of the aquifer, respectively, and M (u, $\beta$) denotes an infinite integral (see list of symbols), sufficient values of which have been tabulated[3] for a practical range of the parameters involved. A more detailed table of this function is available.[4]

Depending on the geometry of the flow system, Eq. 1 may take the form

$$s = c\ M(u, \beta) \quad \ldots\ldots\ldots\ldots (4)$$

in which case the equation is valid for $t < \left[\frac{(2b = r\beta)^2 S_s}{20 K}\right]$, c and $\beta$ being constants that depend on the parameters of the flow system under consideration. For example, if the pumped well is screened throughout its depth of penetration (d = 0), and if the piezometer is of a penetration depth equal to that of the pumped well (z = l), Eq. 1 will reduce to

$$s = \frac{Q}{8 \pi K l} M\left(u, \frac{2 l}{r}\right) \quad \ldots\ldots\ldots\ldots (5a)$$

If, on the other hand, the piezometer is of zero penetration (z = 0), the drawdown equation becomes

$$s = \frac{Q}{4 \pi K l} M\left(u, \frac{l}{r}\right) \quad \ldots\ldots\ldots\ldots (5b)$$

Another example is that of a flow system in which l = 3d and z = d, in which case the equation of drawdown is

$$s = \frac{3 Q}{16 \pi K l} M\left(u, \frac{4 l}{3 r}\right) \quad \ldots\ldots\ldots\ldots (5c)$$

For relatively long periods of pumping, the equation of drawdown for $t > \frac{(b^2 S_s)}{2K}$ has been shown to[5] be

$$s = \frac{Q}{4 \pi K b} \left\{ W(u) + f_s\left(\frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}\right)\right\} \quad \ldots\ldots\ldots (6)$$

[4] Professional Paper 102, Research Div., New Mexico Inst. of Mining and Tech., Socorro, N. Mex.

[5] "Nonsteady Flow to a Well Partially Penetrating an Infinite Leaky Aquifer," by M. S. Hantush, Proceedings, Iraqi Scientific Soc., 1957, p. 10; also reprinted by New Mexico Inst. of Mining and Tech., Socorro, N. Mex.

in which

$$f_s = \frac{4b}{\pi(l - d)} \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) K_o \left(\frac{n \pi r}{b}\right) \cos \left(\frac{n \pi z}{b}\right) \left[\sin \left(\frac{n \pi l}{b}\right) - \sin \left(\frac{n \pi d}{b}\right)\right] \quad \ldots\ldots (7)$$

$K_o$ is the zero-order modified Bessel function of the second kind, and W (u) is the well function for nonleaky aquifers, or what in the mathematical literature is known as the negative exponential integral of (-u). The function is available in tabular form.[6]

*Drawdown in Observation Wells.*—Unlike a piezometer that registers the drawdown at a point in the aquifer, an observation well screened in the aquifer will reflect the average drawdown in the aquifer profile that is in contact with the well screen (or perforated casing). For an observation well that is screened between the depths d' and l' the required equations are as follows:

An approximate equation of average drawdown for $t < \dfrac{\left[2b - \left(\frac{1}{2}\right)(2l + l' + d')\right]^2 S_s}{20\ K}$

that gives results sufficiently accurate for practical application, provided that $(l'/l) < 2$ is

$$\bar{s} \approx \frac{Q}{8 \pi K (l - d)} \bar{E} \left(u, \frac{l}{r}, \frac{d}{r}, \frac{l'}{r}, \frac{d'}{r}\right) \quad \ldots\ldots (8)$$

in which $\bar{s}$ is the average drawdown in the observation well and $\bar{E}$ is the value of the function E of Eq. 1, in which the value of z is replaced by $(l' + d')/2$. An exact solution is available.[3]

If $(r/l) > 1$ and $(l'/l) < 1$, the average drawdown in the observation well can, for all practical purposes, be taken as that given by Eq. 1, with the value of z arbitrarily chosen between l' and zero. The choice is generally made so as to simplify the equation which, in certain cases, may take the form of Eq. 2 and which is valid in the same time interval as that of the latter. The simplified equation would have the form

$$\bar{s} \approx c\ M\ (u, \beta) \quad \ldots\ldots (9)$$

For example, if l = 3 d, a choice of z = d will reduce Eq. 1 to Eq. 5c. Also, if d = 0, a choice of z = 0 will result in Eq. 5b, whereas a choice of z = l gives Eq. 5a. All these equations are of the type of Eq. 9.

For a relatively long period of pumping, the equation of average drawdown for $t > \left(\dfrac{b^2 S_s}{2K}\right)$ is

$$\bar{s} = \frac{Q}{4 \pi K b} \left\{ W(u) + \bar{f}_s \left(\frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{l'}{b}, \frac{d'}{b}\right) \right\} \quad \ldots\ldots (10)$$

6 "Methods for Determining Permeability of Water-Bearing Materials," by L. K. Wenzel, U. S. Geol. Survey, Water-Supply Paper No. 887, 1942, p. 88; also "Hydrology," by C. O. Wisler and E. F. Brater, John Wiley and Sons, New York, N. Y., 1951.



in which

$$\bar{f}_s = \frac{4 b^2}{\pi^2 (l - d)(l' - d')} \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right) K_o \left(\frac{n \pi r}{b}\right) \left[\sin \left(\frac{n \pi l}{b}\right) - \sin \left(\frac{n \pi d}{b}\right)\right] \left[\sin \left(\frac{n \pi l'}{b}\right) - \sin \left(\frac{n \pi d'}{b}\right)\right] \quad \ldots\ldots (11)$$

Eq. 11, as well as Eq. 6, shows that in this range of time, the rate of change of drawdown is the same as though the pumped well completely penetrated the aquifer. In other words, the effect of partial penetration on the drawdown has attained its maximum value.

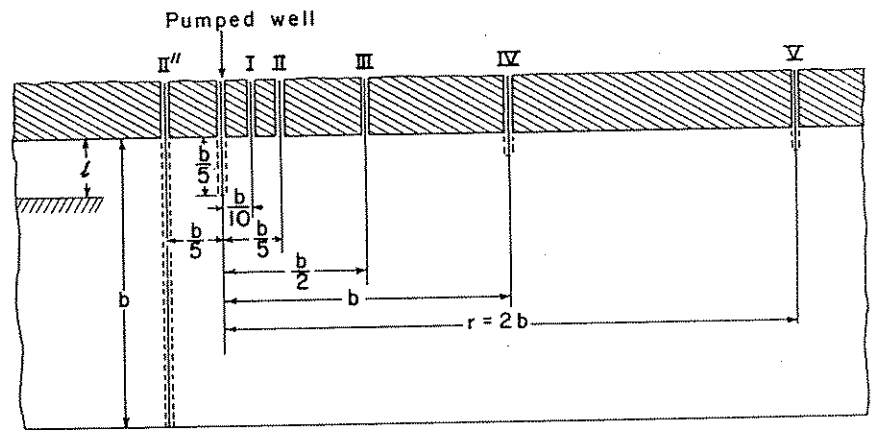


FIG. 2.—DIAGRAMMATIC REPRESENTATION OF PARTIALLY PENETRATING WELLS

*Drawdown in Piezometers or Wells for (r/b) > 1.5.*—For relatively large distances, that is $(r/b) > 1.5$, the equation of drawdown has been shown[7] to be

$$s = \frac{Q}{4 \pi K b} W(u) \quad \ldots\ldots (12)$$

In fact, Eq. 12 gives results sufficiently accurate for practical purposes even for (r/b) as small as one, provided $u < 0.1\ (r/b)^2$. The equation is the same as it would be if the pumped well completely penetrated the aquifer (Theis formula).[8]

7 "Nonsteady Flow to a Well Partially Penetrating an Infinite Leaky Aquifer," by M. S. Hantush, Proceedings, Iraqi Scientific Soc., 1957, p. 10; also reprinted by New Mexico Inst. of Mining and Tech., Socorro, N. Mex.

8 "Groundwater Hydrology," by David K. Todd, John Wiley and Sons, Inc., New York, 1959, p. 90; also "Arid Zone Hydrology Recent Developments" by H. Schoeller, UNESCO, Paris, France, 1959, p. 37.

*Drawdown in the Pumping Well.*—The equations of drawdown that have been presented in the previous sections are derived on the assumption that the flux entering the pumped well is uniformly distributed along the water entry face of the well. Theoretically speaking the hydraulic head, rather than the flux, along the face just outside of the pumped well screen should be uniform. This uniform head distribution can be achieved by a lengthy process involving the distribution of varying flux elements along the well axis. The drawdown at points not in the immediate vicinity of the pumped well, obtained by the simpler equations that are based on a uniform flux across the well screen, does not deviate appreciably from that obtained by the assumedly more exact, but lengthy process. In the actual problem neither a uniform flux nor a uniform head is really conceived along the face of the well, due to several involved field and operational conditions. The drawdown, in this case, will have a value between the two theoretical extremes. Thus, each of the two theoretical expressions can be used to very closely represent the actual case. The simpler equations are, of course, more appealing.

The theoretical drawdown equations, derived on the assumption of uniform flux along the well screen, will obviously give a variable head distribution along the face of the pumped well. It has been shown,[9] however, that the maximum drawdown (least hydraulic head) at the face of the well that these equations give, is very closely equal to that which obtains if the hydraulic head along the face of the well is maintained uniform. The point along the face of the well at which the least hydraulic head occurs depends on the space position of the well screen. If the well is screened throughout its depth of penetration, the least head takes place at the top of the aquifer ($z = 0$). If, on the other hand, the well completely penetrates the aquifer and only its lower part is screened, the least head occurs at the bottom of the aquifer ($z = b$). But if the well is screened between the depths d and l, the least head develops at a depth somewhere between l and d, being closer to d if most of the well screen is in the upper half of the aquifer, and closer to l if the reverse is true. Computation in the drawdown equation shows, however, that the value of maximum drawdown does not differ appreciably from that obtained for $z = (1/2)(l + d)$. Consequently, if so-called well losses are excluded, the water level in a pumping well can be calculated from the equations of drawdown in piezometers (presented in the previous sections) by substituting therein $r_W$ for r, and the value of z at which the least hydraulic head takes place, $r_W$ being the effective radius of the well.

## PROPERTIES OF DRAWDOWN EQUATIONS

Some characteristic features of the drawdown equations that are of importance in graphical computations for the formation coefficients will be given.

*Equations for Special Cases.*—Equations of drawdown in piezometers or observation wells that, during relatively short periods of pumping $\left(t < \left[(2b - r\beta)^2 S_S/20K\right]\right)$, are of the type of Eq. 4 or Eq. 9, are here called "equations for special cases." Fig. 3 gives theoretical graphs of drawdown versus the

9 "The Flow of Homogeneous Fluids Through Porous Media," by M. Muskat, McGraw-Hill Book Co., Inc., New York, N. Y., 1937; or J. W. Edwards, Inc., Ann Arbor, Mich., 1946.



logarithm of time for such equations. These graphs have the following properties:

1. In the period $t < \left[(r\beta)^2 S_S/20K\right]$, the drawdown is given by

$$s = c\,W(u) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

This follows from the value of the function $M(u, \beta)$ for large values of u; that is, small values of time [the approximation of $M(u, \beta)$ has been presented by the writer[3]].

Eq. 13 indicates that in the initial period of pumping, the flow behaves as though the aquifer ended at the bottom of the pumping well. This is shown by the dashed curves of Fig. 3 (I' and II'), that give the time-drawdown variation for wells I and II, respectively, if the aquifer is assumed to end at the bottom of the pumped well. Except for the pair of curves I and I', in the example of Fig. 3 (prepared for assumed values of the parameters), the other pairs (II and

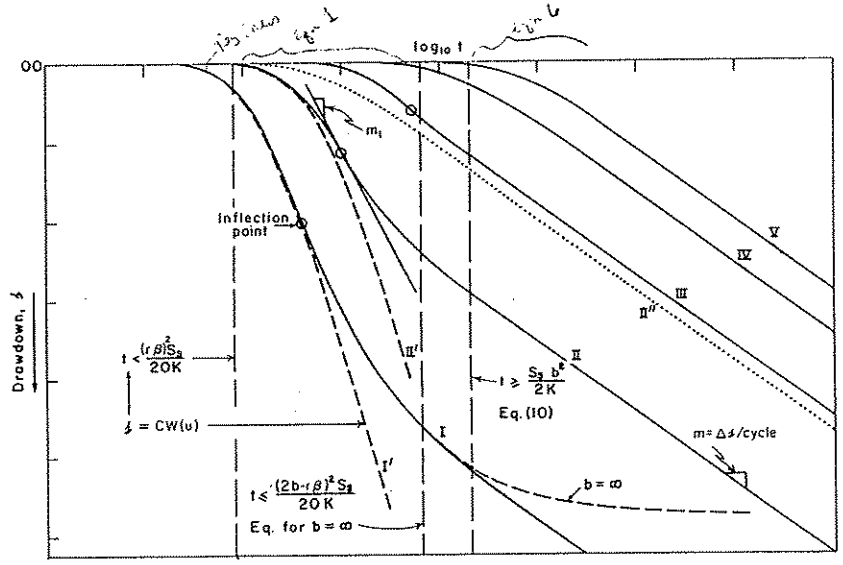


FIG. 3.—TIME-DRAWDOWN VARIATION DUE TO A STEADY WELL PARTIALLY PENETRATING AN ARTESIAN AQUIFER

II', and others not shown) fall outside the period mentioned above for any measurable drawdown. The curves of each pair deviate from each other, although the general trend of variation prior to the formation of the inflection point appears to be the same, the deviation being greater for larger distances from the pumped well.

2. In general, the curves have an inflection point that develops within the period $t < \left[(2b - r\beta)^2 S_S/20K\right]$. This point is located by the relation

$$\frac{\beta^2}{\sqrt{\pi}} = f(x)$$

$$= x\,e^{x^2}\,\mathrm{erf}(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

TABLE 1.—VALUES OF THE FUNCTION erf(x) & $f(x) = xe^{x^2}\,\mathrm{erf}(x)$

| x | erf(x) | f(x) | x | erf(x) | f(x) | x | erf(x) | f(x) | x | erf(x) | f(x) | x | erf(x) | f(x) | x | erf(x) | f(x) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.005 | 0.0056 | 2821-7) | 0.505 | 0.5245 | 0.34207 | 1.005 | 0.8448 | 2.3078 | 1.505 | 0.9667 | 14.012 | 2.005 | 0.9954 | 133.12 | 2.505 | 0.9996 | 1310.0 |
| 0.010 | 0.0113 | 2131-6) | 0.510 | 0.5292 | 0.35007 | 1.010 | 0.8468 | 2.3721 | 1.510 | 0.9671 | 14.282 | 2.010 | 0.9955 | 113.72 | 2.510 | 0.9996 | 1364.5 |
| 0.015 | 0.0169 | 2541-6) | 0.515 | 0.5336 | 0.35825 | 1.015 | 0.8488 | 2.4338 | 1.515 | 0.9675 | 14.554 | 2.015 | 0.9956 | 116.33 | 2.515 | 0.9996 | 1404.0 |
| 0.020 | 0.0226 | 4511-6) | 0.520 | 0.5379 | 0.36656 | 1.020 | 0.8508 | 2.4811 | 1.520 | 0.9684 | 14.835 | 2.020 | 0.9957 | 119.01 | 2.520 | 0.9996 | 1442.7 |
| 0.025 | 0.0282 | 7051-6) | 0.525 | 0.5422 | 0.37497 | 1.025 | 0.8528 | 2.4995 | 1.525 | 0.9690 | 15.181 | 2.025 | 0.9958 | 121.75 | 2.525 | 0.9996 | 1482.5 |
| 0.030 | 0.0338 | 0.00102 | 0.530 | 0.5465 | 0.38357 | 1.030 | 0.8548 | 2.5435 | 1.530 | 0.9695 | 15.434 | 2.030 | 0.9959 | 124.57 | 2.530 | 0.9997 | 1523.5 |
| 0.035 | 0.0395 | 0.00138 | 0.535 | 0.5507 | 0.39227 | 1.035 | 0.8567 | 2.5882 | 1.535 | 0.9701 | 15.711 | 2.035 | 0.9960 | 127.45 | 2.535 | 0.9997 | 1565.7 |
| 0.040 | 0.0451 | 0.00181 | 0.540 | 0.5549 | 0.40114 | 1.040 | 0.8587 | 2.6338 | 1.540 | 0.9706 | 16.015 | 2.040 | 0.9961 | 130.40 | 2.540 | 0.9997 | 1609.1 |
| 0.045 | 0.0507 | 0.00204 | 0.545 | 0.5590 | 0.41011 | 1.045 | 0.8606 | 2.6800 | 1.545 | 0.9712 | 16.326 | 2.045 | 0.9962 | 133.43 | 2.545 | 0.9997 | 1653.8 |
| 0.050 | 0.0564 | 0.00283 | 0.550 | 0.5633 | 0.41929 | 1.050 | 0.8626 | 2.7264 | 1.550 | 0.9716 | 16.642 | 2.050 | 0.9963 | 136.54 | 2.550 | 0.9997 | 1699.6 |
| 0.055 | 0.0620 | 0.00342 | 0.555 | 0.5675 | 0.42855 | 1.055 | 0.8645 | 2.7752 | 1.555 | 0.9722 | 16.965 | 2.055 | 0.9963 | 139.72 | 2.555 | 0.9997 | 1747.2 |
| 0.060 | 0.0676 | 0.00407 | 0.560 | 0.5716 | 0.43801 | 1.060 | 0.8661 | 2.8240 | 1.560 | 0.9726 | 17.297 | 2.060 | 0.9964 | 142.98 | 2.560 | 0.9997 | 1796.0 |
| 0.065 | 0.0732 | 0.00478 | 0.565 | 0.5757 | 0.44763 | 1.065 | 0.8680 | 2.8737 | 1.565 | 0.9731 | 17.634 | 2.065 | 0.9965 | 146.32 | 2.565 | 0.9997 | 1846.2 |
| 0.070 | 0.0789 | 0.00558 | 0.570 | 0.5798 | 0.45738 | 1.070 | 0.8698 | 2.9242 | 1.570 | 0.9736 | 17.979 | 2.070 | 0.9966 | 149.76 | 2.570 | 0.9997 | 1897.9 |
| 0.075 | 0.0845 | 0.00637 | 0.575 | 0.5839 | 0.46727 | 1.075 | 0.8716 | 2.9756 | 1.575 | 0.9741 | 18.330 | 2.075 | 0.9967 | 153.27 | 2.575 | 0.9997 | 1951.2 |
| 0.080 | 0.0901 | 0.00725 | 0.580 | 0.5879 | 0.47735 | 1.080 | 0.8733 | 3.0280 | 1.580 | 0.9746 | 18.691 | 2.080 | 0.9967 | 156.88 | 2.580 | 0.9997 | 2006.0 |
| 0.085 | 0.0957 | 0.00819 | 0.585 | 0.5919 | 0.48760 | 1.085 | 0.8751 | 3.0813 | 1.585 | 0.9750 | 19.058 | 2.085 | 0.9968 | 160.58 | 2.585 | 0.9998 | 2062.6 |
| 0.090 | 0.1013 | 0.00919 | 0.590 | 0.5959 | 0.49797 | 1.090 | 0.8768 | 3.1335 | 1.590 | 0.9755 | 19.433 | 2.090 | 0.9969 | 164.37 | 2.590 | 0.9998 | 2120.7 |
| 0.095 | 0.1069 | 0.01026 | 0.595 | 0.5999 | 0.50858 | 1.095 | 0.8785 | 3.1907 | 1.595 | 0.9759 | 19.815 | 2.095 | 0.9970 | 168.26 | 2.595 | 0.9998 | 2180.6 |
| 0.100 | 0.1125 | 0.01136 | 0.600 | 0.6039 | 0.51931 | 1.100 | 0.8802 | 3.2470 | 1.600 | 0.9764 | 20.208 | 2.100 | 0.9970 | 172.25 | 2.600 | 0.9998 | 2242.3 |
| 0.105 | 0.1181 | 0.01253 | 0.605 | 0.6078 | 0.53023 | 1.105 | 0.8819 | 3.3061 | 1.605 | 0.9768 | 20.606 | 2.105 | 0.9971 | 176.34 | 2.605 | 0.9998 | 2305.9 |
| 0.110 | 0.1236 | 0.01384 | 0.610 | 0.6117 | 0.54132 | 1.110 | 0.8835 | 3.3683 | 1.610 | 0.9772 | 21.015 | 2.110 | 0.9972 | 180.54 | 2.610 | 0.9998 | 2371.4 |
| 0.115 | 0.1292 | 0.01525 | 0.615 | 0.6156 | 0.55260 | 1.115 | 0.8852 | 3.4215 | 1.615 | 0.9776 | 21.433 | 2.115 | 0.9972 | 184.84 | 2.615 | 0.9998 | 2438.8 |
| 0.120 | 0.1348 | 0.01640 | 0.620 | 0.6194 | 0.56402 | 1.120 | 0.8868 | 3.4819 | 1.620 | 0.9780 | 21.860 | 2.120 | 0.9973 | 189.25 | 2.620 | 0.9998 | 2508.5 |
| 0.125 | 0.1403 | 0.01782 | 0.625 | 0.6232 | 0.57568 | 1.125 | 0.8884 | 3.5432 | 1.625 | 0.9784 | 22.293 | 2.125 | 0.9973 | 193.78 | 2.625 | 0.9998 | 2579.9 |
| 0.130 | 0.1459 | 0.01928 | 0.630 | 0.6271 | 0.58754 | 1.130 | 0.8900 | 3.6058 | 1.630 | 0.9788 | 22.739 | 2.130 | 0.9974 | 198.43 | 2.630 | 0.9998 | 2653.6 |
| 0.135 | 0.1514 | 0.02082 | 0.635 | 0.6308 | 0.59950 | 1.135 | 0.8915 | 3.6693 | 1.635 | 0.9792 | 23.191 | 2.135 | 0.9975 | 203.17 | 2.635 | 0.9998 | 2731.8 |
| 0.140 | 0.1570 | 0.02241 | 0.640 | 0.6346 | 0.61173 | 1.140 | 0.8931 | 3.7362 | 1.640 | 0.9796 | 23.658 | 2.140 | 0.9975 | 208.06 | 2.640 | 0.9998 | 2807.8 |
| 0.145 | 0.1625 | 0.02406 | 0.645 | 0.6383 | 0.62412 | 1.145 | 0.8946 | 3.8002 | 1.645 | 0.9800 | 24.132 | 2.145 | 0.9976 | 213.09 | 2.645 | 0.9998 | 2888.5 |
| 0.150 | 0.1680 | 0.02577 | 0.650 | 0.6420 | 0.63675 | 1.150 | 0.8961 | 3.8674 | 1.650 | 0.9804 | 24.618 | 2.150 | 0.9976 | 218.24 | 2.650 | 0.9998 | 2971.6 |
| 0.155 | 0.1735 | 0.02755 | 0.655 | 0.6457 | 0.64952 | 1.155 | 0.8976 | 3.9357 | 1.655 | 0.9808 | 25.113 | 2.155 | 0.9977 | 223.52 | 2.655 | 0.9998 | 3057.3 |
| 0.160 | 0.1790 | 0.02938 | 0.660 | 0.6494 | 0.66256 | 1.160 | 0.8991 | 4.0055 | 1.660 | 0.9811 | 25.619 | 2.160 | 0.9978 | 228.94 | 2.660 | 0.9998 | 3145.5 |
| 0.165 | 0.1845 | 0.03128 | 0.665 | 0.6530 | 0.67574 | 1.165 | 0.9006 | 4.0763 | 1.665 | 0.9815 | 26.137 | 2.165 | 0.9978 | 234.49 | 2.665 | 0.9998 | 3236.5 |
| 0.170 | 0.1900 | 0.03325 | 0.670 | 0.6566 | 0.68921 | 1.170 | 0.9020 | 4.1485 | 1.670 | 0.9818 | 26.666 | 2.170 | 0.9979 | 240.20 | 2.670 | 0.9998 | 3330.2 |
| 0.175 | 0.1955 | 0.03527 | 0.675 | 0.6602 | 0.70288 | 1.175 | 0.9036 | 4.2220 | 1.675 | 0.9822 | 27.205 | 2.175 | 0.9979 | 246.05 | 2.675 | 0.9999 | 3426.9 |
| 0.180 | 0.2009 | 0.03736 | 0.680 | 0.6638 | 0.71669 | 1.180 | 0.9048 | 4.2970 | 1.680 | 0.9825 | 27.758 | 2.180 | 0.9980 | 252.06 | 2.680 | 0.9999 | 3526.5 |
| 0.185 | 0.2064 | 0.03951 | 0.685 | 0.6673 | 0.73079 | 1.185 | 0.9062 | 4.3732 | 1.685 | 0.9828 | 28.322 | 2.185 | 0.9980 | 258.22 | 2.685 | 0.9999 | 3629.1 |
| 0.190 | 0.2118 | 0.04171 | 0.690 | 0.6708 | 0.74514 | 1.190 | 0.9076 | 4.4509 | 1.690 | 0.9832 | 28.900 | 2.190 | 0.9981 | 264.55 | 2.690 | 0.9999 | 3734.9 |
| 0.195 | 0.2173 | 0.04401 | 0.695 | 0.6743 | 0.75965 | 1.195 | 0.9090 | 4.5300 | 1.695 | 0.9835 | 29.489 | 2.195 | 0.9981 | 271.04 | 2.695 | 0.9999 | 3844.0 |
| 0.200 | 0.2227 | 0.04636 | 0.700 | 0.6778 | 0.77446 | 1.200 | 0.9103 | 4.6106 | 1.700 | 0.9838 | 30.092 | 2.200 | 0.9981 | 277.71 | 2.700 | 0.9999 | 3956.5 |
| 0.205 | 0.2281 | 0.04877 | 0.705 | 0.6813 | 0.78949 | 1.205 | 0.9116 | 4.6925 | 1.705 | 0.9841 | 30.709 | 2.205 | 0.9982 | 284.55 | 2.705 | 0.9999 | 4072.4 |
| 0.210 | 0.2335 | 0.05125 | 0.710 | 0.6847 | 0.80477 | 1.210 | 0.9127 | 4.7748 | 1.710 | 0.9844 | 31.340 | 2.210 | 0.9982 | 291.58 | 2.710 | 0.9999 | 4191.9 |
| 0.215 | 0.2389 | 0.05380 | 0.715 | 0.6881 | 0.82025 | 1.215 | 0.9143 | 4.8611 | 1.715 | 0.9847 | 31.983 | 2.215 | 0.9983 | 298.78 | 2.715 | 0.9999 | 4315.1 |
| 0.220 | 0.2443 | 0.05642 | 0.720 | 0.6914 | 0.83606 | 1.220 | 0.9155 | 4.9481 | 1.720 | 0.9850 | 32.642 | 2.220 | 0.9983 | 306.19 | 2.720 | 0.9999 | 4442.2 |
| 0.225 | 0.2497 | 0.05909 | 0.725 | 0.6948 | 0.85203 | 1.225 | 0.9168 | 5.0365 | 1.725 | 0.9853 | 33.315 | 2.225 | 0.9984 | 313.78 | 2.725 | 0.9999 | 4573.2 |
| 0.230 | 0.2550 | 0.06183 | 0.730 | 0.6981 | 0.86827 | 1.230 | 0.9181 | 5.1261 | 1.730 | 0.9856 | 34.004 | 2.230 | 0.9984 | 321.59 | 2.730 | 0.9999 | 4708.5 |
| 0.235 | 0.2604 | 0.06466 | 0.735 | 0.7014 | 0.88480 | 1.235 | 0.9193 | 5.2180 | 1.735 | 0.9859 | 34.707 | 2.235 | 0.9984 | 330.12 | 2.735 | 0.9999 | 4847.6 |
| 0.240 | 0.2657 | 0.06755 | 0.740 | 0.7047 | 0.90166 | 1.240 | 0.9205 | 5.3115 | 1.740 | 0.9861 | 35.429 | 2.240 | 0.9985 | 337.82 | 2.740 | 0.9999 | 4991.2 |
| 0.245 | 0.2710 | 0.07050 | 0.745 | 0.7079 | 0.91869 | 1.245 | 0.9217 | 5.4060 | 1.745 | 0.9864 | 36.164 | 2.245 | 0.9985 | 346.27 | 2.745 | 0.9999 | 5139.4 |
| 0.250 | 0.2763 | 0.07354 | 0.750 | 0.7112 | 0.93612 | 1.250 | 0.9229 | 5.5033 | 1.750 | 0.9867 | 36.918 | 2.250 | 0.9985 | 354.54 | 2.750 | 0.9999 | 5292.2 |
| 0.255 | 0.2816 | 0.07666 | 0.755 | 0.7144 | 0.95372 | 1.255 | 0.9241 | 5.6026 | 1.755 | 0.9870 | 37.688 | 2.255 | 0.9986 | 363.85 | | | |
| 0.260 | 0.2869 | 0.07981 | 0.760 | 0.7175 | 0.97161 | 1.260 | 0.9252 | 5.7031 | 1.760 | 0.9872 | 38.476 | 2.260 | 0.9986 | 373.51 | | | |
| 0.265 | 0.2922 | 0.08305 | 0.765 | 0.7207 | 0.98985 | 1.265 | 0.9264 | 5.8055 | 1.765 | 0.9874 | 39.279 | 2.265 | 0.9986 | 382.38 | | | |
| 0.270 | 0.2974 | 0.08637 | 0.770 | 0.7238 | 1.0083 | 1.270 | 0.9275 | 5.9103 | 1.770 | 0.9877 | 40.105 | 2.270 | 0.9987 | 392.03 | | | |
| 0.275 | 0.3027 | 0.08977 | 0.775 | 0.7269 | 1.0271 | 1.275 | 0.9286 | 6.0159 | 1.775 | 0.9879 | 40.946 | 2.275 | 0.9987 | 401.94 | | | |
| 0.280 | 0.3078 | 0.09324 | 0.780 | 0.7300 | 1.0464 | 1.280 | 0.9297 | 6.1252 | 1.780 | 0.9882 | 41.809 | 2.280 | 0.9987 | 412.12 | | | |
| 0.285 | 0.3131 | 0.09678 | 0.785 | 0.7331 | 1.0657 | 1.285 | 0.9308 | 6.2353 | 1.785 | 0.9884 | 42.689 | 2.285 | 0.9988 | 422.57 | | | |
| 0.290 | 0.3183 | 0.10039 | 0.790 | 0.7361 | 1.0853 | 1.290 | 0.9319 | 6.3481 | 1.790 | 0.9886 | 43.593 | 2.290 | 0.9988 | 433.30 | | | |
| 0.295 | 0.3235 | 0.10409 | 0.795 | 0.7391 | 1.1055 | 1.295 | 0.9330 | 6.4626 | 1.795 | 0.9889 | 44.514 | 2.295 | 0.9988 | 444.31 | | | |
| 0.300 | 0.3286 | 0.10788 | 0.800 | 0.7421 | 1.1259 | 1.300 | 0.9340 | 6.5798 | 1.800 | 0.9891 | 45.460 | 2.300 | 0.9989 | 455.66 | | | |
| 0.305 | 0.3338 | 0.11173 | 0.805 | 0.7451 | 1.1466 | 1.305 | 0.9350 | 6.6991 | 1.805 | 0.9893 | 46.423 | 2.305 | 0.9989 | 467.31 | | | |
| 0.310 | 0.3389 | 0.11566 | 0.810 | 0.7480 | 1.1677 | 1.310 | 0.9361 | 6.8211 | 1.810 | 0.9895 | 47.413 | 2.310 | 0.9989 | 479.26 | | | |
| 0.315 | 0.3440 | 0.11967 | 0.815 | 0.7509 | 1.1891 | 1.315 | 0.9371 | 6.9452 | 1.815 | 0.9897 | 48.422 | 2.315 | 0.9989 | 491.55 | | | |
| 0.320 | 0.3491 | 0.12376 | 0.820 | 0.7538 | 1.2108 | 1.320 | 0.9381 | 7.0717 | 1.820 | 0.9899 | 49.458 | 2.320 | 0.9990 | 504.18 | | | |
| 0.325 | 0.3542 | 0.12794 | 0.825 | 0.7567 | 1.2330 | 1.325 | 0.9391 | 7.1999 | 1.825 | 0.9901 | 50.514 | 2.325 | 0.9990 | 517.15 | | | |
| 0.330 | 0.3593 | 0.13221 | 0.830 | 0.7595 | 1.2554 | 1.330 | 0.9400 | 7.3318 | 1.830 | 0.9904 | 51.600 | 2.330 | 0.9990 | 530.48 | | | |
| 0.335 | 0.3643 | 0.13654 | 0.835 | 0.7623 | 1.2783 | 1.335 | 0.9410 | 7.4658 | 1.835 | 0.9906 | 52.706 | 2.335 | 0.9990 | 544.17 | | | |
| 0.340 | 0.3694 | 0.14098 | 0.840 | 0.7651 | 1.3015 | 1.340 | 0.9419 | 7.6015 | 1.840 | 0.9907 | 53.843 | 2.340 | 0.9991 | 558.25 | | | |
| 0.345 | 0.3744 | 0.14549 | 0.845 | 0.7679 | 1.3251 | 1.345 | 0.9428 | 7.7406 | 1.845 | 0.9909 | 55.002 | 2.345 | 0.9991 | 572.71 | | | |
| 0.350 | 0.3794 | 0.15009 | 0.850 | 0.7707 | 1.3492 | 1.350 | 0.9438 | 7.8827 | 1.850 | 0.9911 | 56.191 | 2.350 | 0.9991 | 590.53 | | | |
| 0.355 | 0.3844 | 0.15477 | 0.855 | 0.7734 | 1.3722 | 1.355 | 0.9447 | 8.0271 | 1.855 | 0.9913 | 57.405 | 2.355 | 0.9991 | 602.86 | | | |
| 0.360 | 0.3893 | 0.15956 | 0.860 | 0.7761 | 1.3970 | 1.360 | 0.9456 | 8.1751 | 1.860 | 0.9915 | 58.651 | 2.360 | 0.9992 | 618.56 | | | |
| 0.365 | 0.3943 | 0.16441 | 0.865 | 0.7788 | 1.4236 | 1.365 | 0.9464 | 8.3252 | 1.865 | 0.9917 | 59.924 | 2.365 | 0.9992 | 634.71 | | | |
| 0.370 | 0.3992 | 0.16938 | 0.870 | 0.7814 | 1.4452 | 1.370 | 0.9473 | 8.4790 | 1.870 | 0.9918 | 61.229 | 2.370 | 0.9992 | 651.30 | | | |
| 0.375 | 0.4041 | 0.17443 | 0.875 | 0.7841 | 1.4751 | 1.375 | 0.9482 | 8.6349 | 1.875 | 0.9920 | 62.563 | 2.375 | 0.9992 | 668.36 | | | |
| 0.380 | 0.4090 | 0.17956 | 0.880 | 0.7867 | 1.5018 | 1.380 | 0.9490 | 8.7948 | 1.880 | 0.9922 | 63.931 | 2.380 | 0.9992 | 685.90 | | | |
| 0.385 | 0.4139 | 0.18479 | 0.885 | 0.7893 | 1.5286 | 1.385 | 0.9499 | 8.9567 | 1.885 | 0.9923 | 65.328 | 2.385 | 0.9993 | 703.92 | | | |
| 0.390 | 0.4187 | 0.19014 | 0.890 | 0.7918 | 1.5561 | 1.390 | 0.9507 | 9.1228 | 1.890 | 0.9925 | 66.762 | 2.390 | 0.9993 | 722.46 | | | |
| 0.395 | 0.4236 | 0.19556 | 0.895 | 0.7944 | 1.5839 | 1.395 | 0.9515 | 9.2925 | 1.895 | 0.9926 | 68.226 | 2.395 | 0.9993 | 741.51 | | | |
| 0.400 | 0.4284 | 0.20109 | 0.900 | 0.7969 | 1.6122 | 1.400 | 0.9523 | 9.4645 | 1.900 | 0.9928 | 69.729 | 2.400 | 0.9993 | 761.12 | | | |
| 0.405 | 0.4332 | 0.20671 | 0.905 | 0.7994 | 1.6410 | 1.405 | 0.9531 | 9.6399 | 1.905 | 0.9929 | 71.264 | 2.405 | 0.9993 | 781.25 | | | |
| 0.410 | 0.4380 | 0.21243 | 0.910 | 0.8019 | 1.6702 | 1.410 | 0.9539 | 9.8203 | 1.910 | 0.9931 | 72.841 | 2.410 | 0.9994 | 801.96 | | | |
| 0.415 | 0.4427 | 0.21826 | 0.915 | 0.8043 | 1.7000 | 1.415 | 0.9546 | 10.003 | 1.915 | 0.9932 | 74.450 | 2.415 | 0.9994 | 823.27 | | | |
| 0.420 | 0.4475 | 0.22419 | 0.920 | 0.8068 | 1.7303 | 1.420 | 0.9554 | 10.190 | 1.920 | 0.9934 | 76.103 | 2.420 | 0.9994 | 845.17 | | | |
| 0.425 | 0.4522 | 0.23023 | 0.925 | 0.8092 | 1.7612 | 1.425 | 0.9561 | 10.380 | 1.925 | 0.9935 | 77.791 | 2.425 | 0.9994 | 867.70 | | | |
| 0.430 | 0.4569 | 0.23636 | 0.930 | 0.8116 | 1.7924 | 1.430 | 0.9569 | 10.575 | 1.930 | 0.9937 | 79.524 | 2.430 | 0.9994 | 890.86 | | | |
| 0.435 | 0.4616 | 0.24259 | 0.935 | 0.8139 | 1.8242 | 1.435 | 0.9576 | 10.773 | 1.935 | 0.9938 | 81.295 | 2.435 | 0.9994 | 914.60 | | | |
| 0.440 | 0.4662 | 0.24896 | 0.940 | 0.8163 | 1.8565 | 1.440 | 0.9583 | 10.976 | 1.940 | 0.9939 | 83.112 | 2.440 | 0.9994 | 939.20 | | | |
| 0.445 | 0.4709 | 0.25542 | 0.945 | 0.8186 | 1.8894 | 1.445 | 0.9590 | 11.182 | 1.945 | 0.9941 | 84.930 | 2.445 | 0.9995 | 964.41 | | | |
| 0.450 | 0.4755 | 0.26201 | 0.950 | 0.8209 | 1.9229 | 1.450 | 0.9597 | 11.393 | 1.950 | 0.9942 | 86.916 | 2.450 | 0.9995 | 990.34 | | | |
| 0.455 | 0.4801 | 0.26868 | 0.955 | 0.8232 | 1.9569 | 1.455 | 0.9604 | 11.606 | 1.955 | 0.9943 | 88.827 | 2.455 | 0.9995 | 1017.0 | | | |
| 0.460 | 0.4847 | 0.27548 | 0.960 | 0.8254 | 1.9915 | 1.460 | 0.9611 | 11.826 | 1.960 | 0.9944 | 90.826 | 2.460 | 0.9995 | 1044.5 | | | |
| 0.465 | 0.4892 | 0.28237 | 0.965 | 0.8277 | 2.0267 | 1.465 | 0.9618 | 12.050 | 1.965 | 0.9946 | 92.873 | 2.465 | 0.9995 | 1072.7 | | | |
| 0.470 | 0.4938 | 0.28943 | 0.970 | 0.8299 | 2.0625 | 1.470 | 0.9624 | 12.278 | 1.970 | 0.9947 | 94.972 | 2.470 | 0.9995 | 1101.7 | | | |
| 0.475 | 0.4983 | 0.29658 | 0.975 | 0.8321 | 2.0990 | 1.475 | 0.9630 | 12.510 | 1.975 | 0.9948 | 97.118 | 2.475 | 0.9995 | 1131.6 | | | |
| 0.480 | 0.5028 | 0.30385 | 0.980 | 0.8342 | 2.1360 | 1.480 | 0.9637 | 12.748 | 1.980 | 0.9949 | 99.322 | 2.480 | 0.9996 | 1162.3 | | | |
| 0.485 | 0.5072 | 0.31124 | 0.985 | 0.8368 | 2.1748 | 1.485 | 0.9643 | 12.990 | 1.985 | 0.9950 | 101.58 | 2.485 | 0.9996 | 1194.0 | | | |
| 0.490 | 0.5117 | 0.31876 | 0.990 | 0.8385 | 2.2121 | 1.490 | 0.9649 | 13.238 | 1.990 | 0.9951 | 103.89 | 2.490 | 0.9996 | 1226.5 | | | |
| 0.495 | 0.5161 | 0.32638 | 0.995 | 0.8406 | 2.2509 | 1.495 | 0.9655 | 13.490 | 1.995 | 0.9952 | 106.26 | 2.495 | 0.9996 | 1260.0 | | | |
| 0.500 | 0.5205 | 0.33414 | 1.000 | 0.8427 | 2.2907 | 1.500 | 0.9661 | 13.750 | 2.000 | 0.9953 | 108.69 | 2.500 | 0.9996 | 1294.5 | | | |



in which

$$x = \beta\sqrt{u_i},\quad u_i = \left(\frac{r^2\,S_s}{4\,K\,t_i}\right) \quad \ldots\ldots\ldots\ldots\ldots (15)$$

and $t_i$ is the value of t at the inflection point. The function f(x) is given in Table 1 for a wide range of x.

The drawdown $s_i$ at the inflection point is given by

$$s_i = c\,M\,(u_i,\,\beta) \quad \ldots\ldots\ldots\ldots\ldots (16)$$

and the slope $m_i$ of the curve at that point is

$$m_i = \frac{\Delta s}{\text{cycle}} = 2.3\ c\ e^{-u_i}\ \mathrm{erf}(x) \quad \ldots\ldots\ldots (17)$$

Eq. 14 is obtained by equating to zero the second derivative of Eq. 4 with respect to $\log_{10}t$, and Eq. 16 is obtained from Eq. 4 by substituting $u_i$ for u. After obtaining the first derivative of Eq. 4 with respect to $\log_{10}t$, $u_i$ is substituted for u to obtain Eq. 17.

3. The drawdown curves depart from those that obtain for an infinitely thick aquifer (as described completely by Eq. 4) at an approximate value of t or u given, respectively, by

$$t_d = \frac{(2\,b\ -\ r\,\beta)^2\ S_s}{20\ K} \quad \ldots\ldots\ldots\ldots\ldots (18a)$$

and

$$\frac{1}{u_d} = \frac{(2\,b\ -\ r\,\beta)^2}{5\ r^2} \quad \ldots\ldots\ldots\ldots\ldots (18b)$$

in which $t_d$ and $u_d$ are, respectively, the values of t and u at the point of departure.

4. The inflection of the curves is clearly defined for small values of (r/b). Depending on the geometry of the flow system, the inflection disappears (see curves IV and V, Fig. 3) for

$$\left[\frac{x^2\ (2\,b\ -\ r\,\beta)^2}{5\ (r\ \beta)^2}\right] \leq 1 \quad \ldots\ldots\ldots\ldots\ldots (19)$$

in which x and $\beta$ are the parameters of Eq. 14. Eq. 19 is obtained by noting that the inflection occurs only if

$$t_i < \left[\frac{(2\,b\ -\ r\,\beta)^2\ S_s}{20\ K}\right] \quad \ldots\ldots\ldots\ldots\ldots (20)$$

5. Regardless of the location of the wells, the curves are straight lines having the same slope at relatively large values of time $\left[t > \left(b^2\,S_s/2K\right)\right]$. The slope m of these lines is given by

$$m = \frac{\Delta s}{\text{cycle}} = 2.3\ \frac{Q}{4\ \pi\ K\ b} \quad \ldots\ldots\ldots\ldots\ldots (21)$$

which is obtained by differentiating Eq. 6 or Eq. 10 with respect to $\log_{10} t$ and noting that u becomes very small for large values of time.

*Other Drawdown Equations.*—For other drawdown equations (equations that, during a relatively short period of pumping, are not of the type of Eq. 4 or Eq. 9), the properties previously examined still apply qualitatively. Quantitatively, however, only Eq. 21 and, provided $(r\ \beta)$ is replaced by $(1/2)\ (2\ l + l' + d')$, Eq. 18 are still applicable. In addition, as the length of the screen of an observation well is increased, other variables being held constant, the curve inflection becomes less distinct regardless of the well location. The curve approached is that of the Theis formula, which is obtained when $(l'/b)$ and $(d'/b)$ become one and zero respectively (see curves II and II', Fig. 3.

## RECOVERY EQUATIONS

If t and t' are the times, reckoned respectively from the commencement and end of pumping, the residual drawdown s' in a piezometer during recovery can be shown to be

$$s' = s\ (t) - s\ (t') \quad \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

Similarly, the average residual drawdown $\bar{s}'$ in an observation well is

$$\bar{s}' = \bar{s}\ (t) - \bar{s}\ (t') \quad \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

in which $t = t_0 + t'$ and $t_0$ is the time at which the pumping has ceased. Thus, the recovery equation corresponding to any of the drawdown equations discussed above can be formulated readily, subject to the same time criteria. For example, if the equation of drawdown is of the type that reduces to that of Eq. 4 during a relatively short period of pumping, the recovery equation can be written as[3]

$$s' = c\left[M\ (u,\ \beta) - M\ (u',\ \beta) + f_2\ (u,\ b/r,\ \beta) - f_2\ (u',\ b/r,\ \beta)\right] \ldots (24)$$

in which u' is the value of u after replacing t by t' and

$$f_2\ (u,\ b/r,\ \beta) = \sum_{n=1}^{\infty} M\left[u, \left(\frac{2\ n\ b}{r} + \beta\right)\right] - M\left[u, \left(\frac{2\ n\ b}{r} - \beta\right)\right] \ldots (25)$$

For $t < \left[\dfrac{(2\ b - r\ \beta)^2\ S_s}{20\ K}\right]$, Eq. 24 can be approximated by

$$s' = c\left[M\ (u,\ \beta) - M\ (u',\ \beta)\right] \ldots\ldots\ldots\ldots (26)$$

For $t_0 > \left(\dfrac{b^2\ S_s}{2\ K}\right)$ and $t' < \left[\dfrac{(2\ b - r\ \beta)^2\ S_s}{20\ K}\right]$, Eq. 17 can be approximated by

$$s' = \frac{Q}{4\ \pi\ K\ b}\left[W\ (u) - f_s\right] - c\,M\ (u',\ \beta) \quad \ldots\ldots (27)$$

If both $t_0$ and t' are greater than $(b^2\ S_s/2\ K)$, Eq. 24, as well as the general recovery equation,[3] become

$$s' = \frac{Q}{4\ \pi\ K\ b}\left[W\ (u) - W\ (u')\right] \ldots\ldots\ldots\ldots (28)$$



The preceding equations give the recovery in piezometers. The recovery in observation wells can be similarly obtained by using the appropriate average drawdown equation. Eq. 28 gives the recovery in both cases.

## PROPERTIES OF RECOVERY EQUATIONS

In applying the recovery equation to the analysis of data from aquifer tests, use is made of some characteristic features of the equation.

*Recovery Equations for Special Cases.*—Of importance is the equation of water level recovery after shutoff in a pumped well that is screened throughout its depth of penetration, or in nearby wells that may be considered as piezometers of zero penetration. In such cases, as well as in others (see section

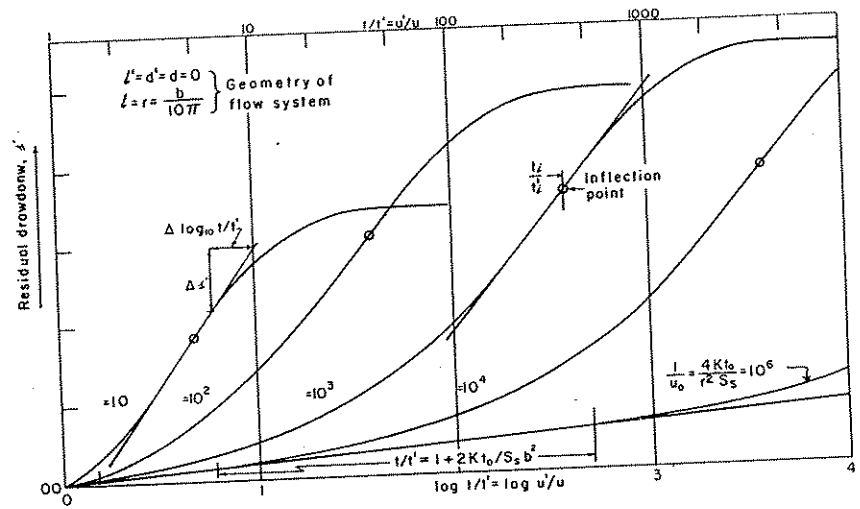


FIG. 4.—LOG (t/t')—RESIDUAL DRAWDOWN VARIATION IN WELLS PARTIALLY PENETRATING AN ARTESIAN AQUIFER

on Equations of Drawdown), the drawdown equation during short periods of pumping of the type of Eqs. 4 or 9. Consequently, the equations given in the preceding section apply. They are here called "recovery equations for special cases."

Fig. 4 represents theoretical plots of the residual drawdown s' (for the set of parameters shown in the figure) versus the logarithm of (t/t'), as given by Eq. 24. These curves have three main characteristics:

1. The curves have a straight portion at their lower left end; that is, a straight line passing through the point of zero residual drawdown and $(t/t') = 1$. This line may be clearly defined, as in the curves $(1/u_0) = (4\ K\ t_0/r^2\ S_s) = 10^6$ and $10^4$; barely recognizable, as in the curve $10^3$; or entirely indistinguishable, as in the curves $10^2$ and 10. The range of time in which this line forms de-

pends on the length of the period of pumping $t_o$, the formation coefficients, and the thickness of the aquifer. That is, the straight line forms in the region for which $(t/t')$ is less than $\left[1 + \left(2\ K\ t_o/b^2\ S_s\right)\right]$. The slope of the straight line, if it forms, is given by

$$m = \frac{\Delta s'}{\text{cycle}} = \frac{2.3\ Q}{4\ \pi\ K\ b} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

which is obtained by differentiating Eq. 28 with respect to $\log_{10}(t/t')$ and observing that u and u' become very small for larger values of time.

2. The curves have, in general, an inflection point, in the neighborhood of which the curve may often be approximated by a straight line, the slope of which for all practical purposes is equal to the slope of the curve at the inflection point. The inflection point forms (if it does) in the period $t' < \left[\frac{(2\ b - r\ \beta)^2\ S_s}{20\ K}\right]$. Thus, the recovery equation in the neighborhood of the inflection point is given by Eq. 24 after dropping out $f_2\left(\frac{u',\ b}{r,\ \beta}\right)$. By differentiating this equation (or any equivalent equation[3]) twice with respect to $\log_{10}(t/t')$ and equating the results to zero, the value of u' at the inflection point, that is the value of $u'_i$, can be shown to be given approximately by the relation

$$\frac{\beta^2}{\sqrt{\pi}} = f(x)$$

$$= x\ e^{x^2}\ \text{erf}(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (30)$$

in which $x = \beta\sqrt{u'_i}$.

The slope of the curve $m_i$ at the inflection point can be obtained from Eq. 24 (or other equivalent form) by differentiating with respect to $\log_{10}(t/t')$ and substituting $u_i$ and $u'_i$ for u and u', respectively, in the resulting expression. This slope can be approximated closely by either of the following two relations, depending on the range of $(t/t')$ in which the inflection point occurs:

a. If the inflection point occurs in the range $(t/t') < 100$, the slope can, for all practical purposes, be taken as

$$m_i = \Delta s'/\text{cycle} = 2.3c \left\{ \left(t_i/t_o\right) \exp\left(-u'_i\right)\ \text{erf}(x) - \left(t_i/t_o\right) \exp\left[-\ (t'/t)_i\ u'_i\right] \text{erf}\left(x\sqrt{(t'/t)_i}\right) \right\} \ldots\ldots\ldots (31)$$

in which $(t'/t)_i$ is the value of $t'/t)$ at the inflection point.

b. If the inflection point occurs in the range $(t/t') > 100$, the second term of Eq. 31 becomes insignificant. The slope is then given by Eq. 31 after neglecting the second term.

3. If the geometry of the flow system is such that the relation

$$\left[x^2\ (2\ b - r\ \beta)^2/5\ (r\ \beta)^2\right] \leq 1 \quad \ldots\ldots\ldots\ldots (32)$$

holds, the curve under consideration will not exhibit an inflection point. The straight portion of the curve that forms (if it does) through the latter part of data will, however, be the same as that described under item 1.



*Other Recovery Equations.*—The properties of the recovery equations for special cases, that were examined in the previous paragraphs, still apply qualitatively for other equations (see the section on Other Drawdown Equations) also. Quantitatively, however, only Eq. 29 is rigorously valid. The other relations can be used only to give an estimate of the values involved therein.

## APPLICATIONS OF THEORY OF AQUIFER TESTS

Before introducing the procedure to be outlined, it should be observed that the effects of partial penetration resemble the effects of leakage from storage in a thick semipervious confining layer.[10] Also, if the curve inflection is apparent, but the period of observation is not long enough to establish the ultimate straight-line variation on a semilogarithmic time-drawdown plot, the effects of partial penetrations resemble the effects of some kind of recharge boundary, such as induced infiltration from beds of streams or lakes,[11] or recharge from water-bearing strata supplying leakage[12] through semipervious confining beds. The same general effects are observed if the wells completely penetrate a sloping water-table aquifer or a wedge-shaped aquifer (nonuniform thickness). The existence of such conditions should be reasonably eliminated prior to performing an analysis, based on the present theory.

*Outline of Methods.*—Semilogarithmic time-drawdown or $\log_{10}(t/t')$—residual drawdown plots or both are an essential part of the analysis of the observed data. Consequently, the first step in the analysis is the construction of such curves.

Analysis of Drawdown Data.—Depending on the geometry of the flow system under consideration, the methods are as follows:

*Inflection Point Method.*—If the equation of drawdown for the observation well under consideration during relatively short periods of pumping is of the type of Eq. 4 or Eq. 9 (or can be approximated by such equations), the following procedure can be followed, provided that the observed semilogarithmic plot clearly indicates the formation of the inflection point within the observed data, and provided also that the tangent of this curve at the inflection can be constructed with approximate accuracy:

(1) On the semilogarithmic plot, construct the tangent in the region of the curve inflection and measure its slope $m_i = \Delta s/\text{cycle}$.

(2) Compute $f(x)$ from $\beta^2/\sqrt{\pi} = f(x)$.

(3) Obtain the value of x and $\text{erf}(x)$ from Table 1 (or from a graph constructed from this table).

(4) Compute $u_i$ from $(x/\beta)^2$ and obtain the value of $\exp(-u_i)$, using a slide rule or appropriate tables.

(5) Compute K, using Eq. 17 with appropriate expression for c.

(6) Obtain the value of $M(u_i, \beta)$, using tabular values of this function and the computed values of $u_i$ and $\beta$. Then compute the value of $s_i$ from Eq. 16.

10 "Modification of the Theory of Leaky Aquifers," by M. S. Hantush, Journal of Geophysical Research, Vol. 65, 1960, p. 3713.

11 "Analysis of Data From Pumping Wells Near a River," by M. S. Hantush, Journal of Geophysical Research, Vol. 64, 1959, p. 1921.

12 "Nonsteady Radial Flow in an Infinite Leaky Aquifer," by M. S. Hantush and C. E. Jacob, Transactions, Amer. Geophysical Union, Vol. 36, 1955, p. 95.

(7) From the semilog plot, obtain the value $t_i$ corresponding to the computed value of $s_i$. Then calculate the specific storage from $S_s = \left(4\ K\ t_i\ u_i / r^2\right)$.

(8) Construct the ultimate straight line of the curve (if it is discernible within the observed data) and measure its slope $m = \Delta s/\text{cycle}$. Then compute T (= Kb) from Eq. 21.

(9) Compute b from the calculated values of K and Kb.

*Type-Curve Method.*—If the semilogarithmic time-drawdown curve of the observed data exhibits an inflection of the type shown in Fig. 3, and if the number and distribution of the observed points are such that the details of the curve prior to the attainment of the ultimate straight-line variation are discernible, a type-curve method can be used to determine the formation coefficients and often the thickness of the aquifer. The method is essentially the "Theis graphical method."[13] In the present case, however, the "type curve," according to the flow system under consideration, is a plot of the E (or $\bar{E}$) function of Eq. 1 (or Eq. 4) versus (1/u) on a logarithmic paper. The observed curve is a plot of s (or $\bar{s}$) versus t on a logarithmic paper of the same scale as that of the type curve. The procedure is as follows:

(1) Construct a type curve for each well, using tabular values of the function $M(u, \beta)$ (or a graph prepared therefrom) and the E (or $\bar{E}$) function of that well.

(2) Plot the observed data on a logarithmic paper of the same scale as that of the type curve.

(3) Superimpose the observed data on the type curve, keeping the coordinate axes of the two curves parallel, and adjust until the best fitting position is obtained. In matching the observed curve to the type curve, one should keep in mind that the observed points for relatively large values of time may deviate upward (having larger values) from the type curve. The deviation is to be expected, because the type curve is for drawdown values during relatively short periods of pumping.

(4) Select an arbitrary point (matching point) anywhere on the two superimposed sheets and record its coordinates; namely,

$$\left[E\ (\text{or}\ \bar{E}),\ 1/u\right] \text{ and } \left[s\ (\text{or}\ \bar{s}),\ t\right]$$

(5) Compute K from Eq. 1 or Eq. 8, whichever applies, and $S_s$ from the relation

$$S_s = \frac{4\ K\ t}{\left(\frac{1}{u}\right) r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (33)$$

(6) If the observed curve departs from the type curve, record the value of (1/u) at the point of "departure." Then compute b from the relations

$$b \approx (1/2)\left[l + z + r\sqrt{5/u_d}\right] \quad \ldots\ldots\ldots\ldots \quad (34a)$$

for piezometer, or

$$b \approx (1/2)\left[l + (1/2)\ (l' + d') + r\sqrt{5/u_d}\right] \quad \ldots\ldots\ldots \quad (34b)$$

13 "Groundwater Hydrology," by David K. Todd, John Wiley and Sons, Inc., New York, 1959, p. 90; also "Arid Zone Hydrology, Recent Development," by H. Schoeller, UNESCO, Paris, France, 1959, p. 37.



for observation wells or, in case E (or $\bar{E}$) reduces to the M function of Eq. 4, from

$$b \approx (1/2)\left[r\ \beta + r\sqrt{\frac{5}{u_d}}\right] \quad \ldots\ldots\ldots\ldots\ldots \quad (34c)$$

in which $u_d$ is the value of u at the "departure" point.

(7) From the computed values of K and b, calculate the transmissivity of the aquifer from T = Kb.

(8) If the observed curve does not depart from the type curve within the observed data, record the value of (1/u) of a point in the vicinity of the last observed point. The thickness of the aquifer is then greater than that obtained from the relations of b in step (6) if one uses therein the present value of (1/u).

(9) It is to be observed that whether the data under analysis are obtained from piezometers or observation wells, and regardless of the location of the point of observation, step (8) of the inflection-point method for computing the transmissivity of the aquifer (Kb) always holds. Carry out this step and obtain the value of (Kb = T). The value of T thus obtained serves as a check for the valve obtained in step (7).

The values of T obtained from step (7) and step (9) may not agree closely. This may be because the best fitting position of the two superimposed curves had not been obtained on the first trial, or because the ultimate straight portion of the curve had been drawn either flatter or steeper than necessary, or because the apparent transmissivity of the aquifer (in case of water-table aquifers) had not attained its uniform value within the period of observation. If such agreement is lacking, the procedure is repeated again with the necessary adjustments in mind, so that the difference between the two values of T is either eliminated or reduced to a minimum. If a great difference persists, it is more likely that the apparent transmissivity has not attained a uniform value within the period of observation.

For certain flow systems, both of the preceding methods may apply, in which case they serve to check each other. Generally, the results of the computations do not agree on the first trial. The disagreement may be a result of under- or over-estimating the slope at the curve inflection, as well as the slope of the ultimate straight line; it may also be due to a poor choice of the best matching position, or to all these factors together. A second trial with the necessary adjustments generally results in a satisfactory answer.

*Theis' or Jacob's Method or Both.*—If the semilogarithmic plot of the observed data does not show any inflection despite a relatively long period of pumping, three possibilities may arise: (a) the observation well, although close to the pumped well, is completely penetrating and screened throughout the aquifer. Such information is generally known a priori; (b) the observation well is relatively distant from the pumped well; that is, $(r/b) > 1.5$. To aid in deciding whether this is the case, information about the approximate thickness of the aquifer should be available. Such information can be obtained from previous geological or geophysical studies, or from analysis of data from a nearby well through procedures such as those previously presented, or at least from the range of the thickness given by the relation

$$b < (1/2)\left[l + l' + (2.4\ 1/x)\right] \quad \ldots\ldots\ldots\ldots\ldots \quad (35)$$

in which x is the solution of Eq. 9; (c) if the first and second possibilities are ruled out, the flow parameters are such that the relation

$$(2b - l - l')^2 \; x^2/5l^2 < 1 \quad \ldots\ldots\ldots\ldots\ldots (36)$$

obtains, x being the solution of Eq. 14.

If the first or the second possibility obtains, the method of Theis and, provided the ultimate straight line of the semilogarithmic plot forms, that of Jacob[13,14] can be used for calculating the formation coefficients. If the third possibility pertains, the type-curve method can be used, provided the number and distribution of the observed points on the curved part of the semilogarithmic plot sufficiently define a curve on the logarithmic plot needed for the type-curve method. Alternatively, the following procedure can be used, provided an estimated value of the aquifer thickness is available:

*Jacob's Method Adjusted for Partial Penetration.*—During the time period in which the ultimate semilogarithmic straight line forms, the drawdown is given, depending on the well observed, by either Eq. 6 or Eq. 10. Because the second term of each of these equations is constant with time, it is clear that Jacob's method can be applied if the numerical value of this constant can be obtained. The procedure is as follows:

(1) On the observed semilogarithmic plot, construct the ultimate straight line and extend it to the zero-drawdown axis.

(2) Obtain the slope, ($m = \Delta s/\text{cycle}$) of this line and its time intercept, $t_p$, on the zero-drawdown axis.

(3) Compute $T = K\,b$ from Eq. 21.

(4) Compute the value of $f_s$ (or $\bar{f}_s$) from their appropriate expressions (a few terms of the series involved are generally sufficient).

(5) Compute $\exp(f_s)$ [or $\exp(\bar{f}_s)$], using a slide rule or appropriate tables. Then calculate the storage coefficient from

$$S = \left[\frac{2.25\; T\; t_p \exp(f_s)}{r^2}\right] \quad \ldots\ldots\ldots\ldots\ldots (37)$$

Analysis of Recovery Data.—If the equation of drawdown in the observation well is one that, during short times, reduces to the type of Eq. 4 or Eq. 5, the procedure of analysis is as follows:

(1) Prepare a plot of $s'$ (or $s'$) versus $\log_{10}(t/t')$ and pass the best-fit curve through the observed points.

(2) Construct the tangent to this curve in the region of the curve inflection and measure its slope $m_i = \Delta s'/\text{cycle}$.

(3) Locate the inflection point on this tangent by inspection and obtain its value of $(t/t')$, which hereafter is designated by $(t/t')_i$. Over or underestimating the location of the inflection point will not materially alter the calculation to follow, as will be pointed out subsequently.

(4) Compute the value of $(t_o/t'_i)$, which is $= (t/t')_i - 1$, and that of $(t/t_o)$, which is $= (t/t')_i/(t_o/t'_i)$.

(5) Compute $f(x)$ from $\beta^2/\sqrt{\pi} = f(x)$ and obtain the corresponding values of x and erf(x) from Table 1.

14 "Hydrology," by C. O. Wisler and E. F. Brater, John Wiley and Sons, New York, 1951, p. 234.



(6) Compute $u'_i$, which is $(x/\beta)^2$, and obtain the value of $\exp(-u_i)$, using a slide rule or a table of the exponential function.

(7) If the value of $(t/t')_i > 100$, calculate K from Eq. 6, neglecting the second term.

(8) If the value of $(t/t')_i < 100$, calculate the values of $u'_i\,(t'/t)_i$ and $x\sqrt{(t'/t)_i}$, and obtain the values of $\exp\left[-u'_i\,(t'/t)_i\right]$ and $\text{erf}\left(x\sqrt{(t'/t)_i}\right)$. Then calculate K from Eq. 31.

(9) From the known values of $t_o$ and $(t'_i/t_o)$, calculate the value of $t'_i$.

(10) Calculate the specific storage from $S_s = (4\,K\,t'_i\,u'_i/r^2)$.

(11) If a straight-line variation between $s'$ and $\log_{10}(t/t')$ is discernible through the latter part of the data, including the origin point, [$(t/t') = 1$, $s' = 0$], then, construct this line and measure its slope $m = \Delta s'/\text{cycle}$; calculate (K b) from Eq. 29; then, knowing the values of K and (K b), compute the value of b.

It is to be observed that if $(t/t')_i > 10$, the second term of Eq. 31 is comparatively very small. Thus, a large error in estimating the value of $(t/t')_i$ will not affect significantly the value of this second term. The largest error, if any will be in the first term of Eq. 31, owing to the factor $(t_i/t_o)$. This error, however, rarely exceeds $\pm$ 3%. Even if one misses the true location of the inflection point by a factor of 10 (a remote possibility) and if, for example, the true value of $(t/t')_i$ is 10 and the estimated value is 100, the error in the value of $(t_i/t_o)$, and consequently in the value of K, will be of the order (1.01 -1.11)/1.11 = -9%. The error in the value of $S_s$ is approximately directly proportional to the error in estimating the value of $t'_i$.

An exact graphical solution for the case in which the short-time drawdown equation is not of the form of Eq. 4, is not available. The procedure outlined previously can be followed to obtain a rough estimation of the formation coefficients. However, if the straight line examined in step (11) forms, the solution for the transmissivity $T = K\,b$, as described in steps (11) and (12), is exact, because in the present case Eq. 29 obtains also.

*Examples of Application.*—Aquifer test data obtained in the Rio Grande Valley near the town of Socorro, N. Mex., will be used to illustrate the application of the methods presented. The wells tested penetrate a water-table aquifer. However, because the observed drawdowns are small compared to the depth of penetration of the pumped wells, the theory presented herein, although presented for artesian conditions, is still fairly applicable to water-table conditions. The computations in what follows were made with a slide rule.

The Faulkner Farm Test.—A schematic representation of the wells tested is shown in Fig. 5. These wells are located in the SW 1/4 NE 1/4 sec. 35, T. 1 S., R. 1 W., Socorro County, N. Mex. Both the pumped well and the observed well are perforated throughout their depths of penetration. These depths are 116 ft and 60 ft, respectively. The observed well is 17 ft distant from the pumped well. An aquifer test using these two wells was run by J. F. Waldron[15] during September 1952. During the test, the pumped well discharged at a uniform rate of 3.57 cfs.

15 "Reconnaissance Geology and Groundwater Study of a Part of Socorro County, New Mexico," by J. F. Waldron, thesis presented to Stanford Univ., at Palo Alto, Calif., in 1956, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

As $(l'/l) = (60/116) < 2$, the average drawdown s in the observed well during relatively short times is given by Eq. 9, which, with $d = d' = 0$, $l' = 60$ ft, $l = 116$ ft, and $r = 17$ ft, becomes

$$\bar{s} = \frac{Q}{928\,\pi\,K}\,\bar{E}\,(u) \quad \ldots\ldots\ldots\ldots\ldots\ldots (38)$$

$$= \frac{Q}{928\,\pi\,K}\left[M\,(u,\,8.6) - M\,(u,\,5.1)\right]$$

Analysis of Drawdown Data.—The observed semilogarithmic curve of Fig. 5 clearly shows an inflection, and because Eq. 24 is not of the form of Eq. 5, the type-curve method is followed.

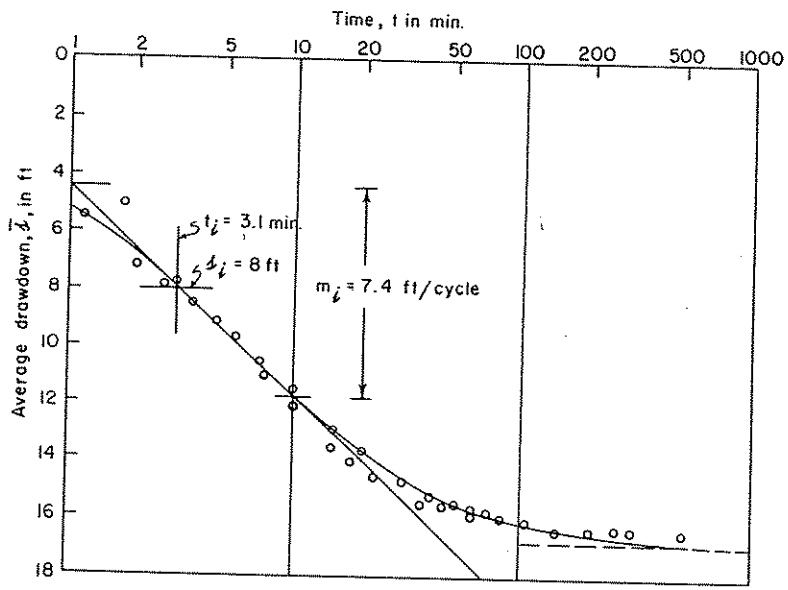


FIG. 5.—TIME-DRAWDOWN VARIATION IN THE FAULKNER TEST OBSERVATION WELL

*Type-Curve Method.*—From Table 1 and $\bar{E}$ (u) of Eq. 38, a type curve is prepared on logarithmic paper, and the observed data are plotted on logarithmic paper of the same scale. The two curves are superimposed and adjusted to obtain the matching position shown in Fig. 6. The valeus $\bar{E}$, $(1/u)$, $\bar{s}$, and t at the matching point shown in the figure are 5.6, 500, 10 ft, and 100 min (6,000 sec), respectively. From Eq. 38, K is computed as $6.86 \times 10^{-4}$ fps, and from $S_s = \left[4\,K\,t/\left(\frac{1}{u}\right) r^2\right]$, $S_s$ is computed as $1.14 \times 10^{-4}$ ft$^{-1}$.

Within the period of the test, the observed curve does not depart from the type curve; that is, the time $t_d$ at the eventual point of departure is greater



than the period of observation. Thus, from Fig. 6, $t_d$ and $(1/u_d)$ are greater than 500 min and 2,500, respectively. Consequently, the thickness of the aquifer is greater than

$$(1/2)\left[l + (1/2)\,l' + r\sqrt{5/u}\right. = (1/2)\left[116 + 30 + 17\sqrt{5 \times 2{,}500}\right] = 1023 \text{ ft}$$

*Inflection-Point Method.*—It has been pointed out that the application of the inflection-point method on data obtained from flow systems for which Eq. 4 or Eq. 9 does not obtain will generally give an approximate solution. This, of course, is due to replacing the actual equation by an approximate equation of the form of Eq. 4. For instance, if the drawdown in the observation well of this

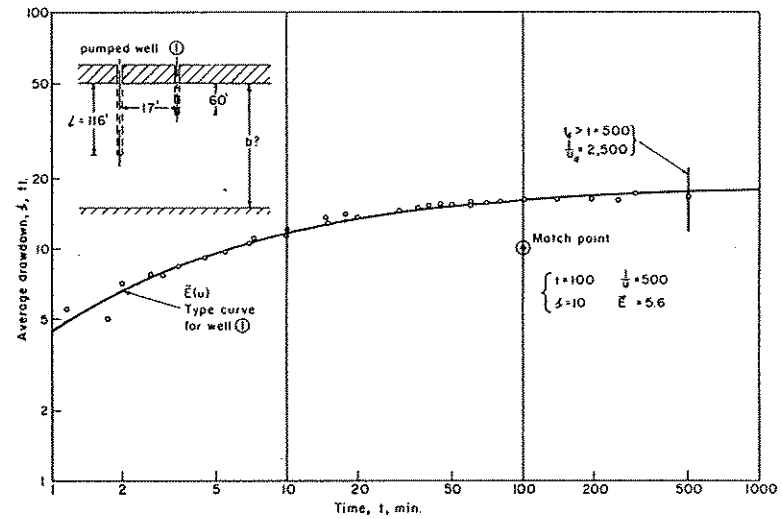


FIG. 6.—OBSERVED DATA AND THE THEORETICAL TYPE CURVE FOR THE OBSERVATION WELL OF THE FAULKNER TEST

example be approximated by that in a piezometer of zero penetration ($l' = 0$), instead of Eq. 38 (the actual equation), the resulting equation is

$$s = \frac{Q}{464\,\pi\,K}\,M\,(u,\,6.82) \quad \ldots\ldots\ldots\ldots\ldots (39)$$

for which $c = \left(\frac{Q}{464\,\pi\,K}\right)$ and $\beta = 6.82$.

Based on Eq. 39, the application of the inflection-point method follows: the semilogarithmic plot of the observed data has been carried out already, as shown in Fig. 5. The tangent to the curve in the region of the curve inflection is constructed; its slope $m_i$ is measured as 7.4 ft per cycle. $f(x) = \beta^2/\sqrt{\pi} = (6.82)^2/\sqrt{\pi} = 26.2$. From Table 1, $x = 1.666$, and $\text{erf}(x) = 0.982$. $u_i = (x/\beta)^2 = 0.06$, and $\exp(-0.06) = 0.94$. From Eq. 17, K is computed as $7 \times 10^{-4}$ fps. In-

terpolation in tables of the function M (u, $\beta$) gives M $(u_i, 6.82) = 2.291$. From Eq. 39, $s_i$ is computed as 8 ft. For $s_i = 8$ ft, Fig. 6 shows $t_i = 3.1$ min = 186 sec. Then, from $S_s = 4\ K\ t_i\ u_i/r^2$, $S_s$ is computed as $1.08 \times 10^{-4}$ ft.

The ultimate straight-line portion of the curve is not discernible within the observed data (see dashed line of Fig. 5). Hence, the transmissivity (Kb) and, consequently, the thickness of the aquifer cannot be determined from the present data.

*The Olson Wells Test.*—Fig. 7 (a) shows semilogarithmic time-drawdown curves for two observation wells prepared from data obtained from a pumping test on what is known as the Olson Wells. The geometry of these wells is shown diagrammatically in Fig. 8. They are located about one-half mile north of the campus of the New Mexico Institute of Mining and Technology, Socorro, N. Mex. Each of the two Olson wells has a 75-ft depth of penetration, of which the bottom 50-ft section is perforated. One of these wells is pumped at a rate of 1.6 cfs. The other well, designated as observation well (1) is 182 ft east of the pumped well. A third well, designated as observation well (2) (known as the Lopez dug well), is 384 ft west of the pumped well and is assumed to be zero penetration (actually about 1.1 ft penetration).

Analysis of drawdown data from well (1)—Eq. 9 will, of course, give the average drawdown in this well. However, because $(r/l) = (182/75) > 1$ and $(l'/l) = (75/75) \leq 1$, Eq. 9 can be approximated by Eq. 1, in which z is chosen arbitrarily between 0 and l (see section on average drawdown for short times). For z = 25 ft (this choice is made because Eq. 1 will reduce to the form of Eq. 4), the average drawdown in this well for small time s will, from Eq. 1, be given by

$$\bar{s} = \frac{Q}{400\ \pi\ K}\ M\ (u, 0.55) \quad \ldots\ldots\ldots\ldots\ldots (40)$$

which is of the form of Eq. 4, with $c = \frac{Q}{400\ \pi\ K}$ and $\beta = 0.55$. Because the inflection of the semilogarithmic curve of the observed data is clearly manifested (see Fig. 7 (a)), the inflection-point method and/or the type-curve method will provide a solution for the formation parameters.

*Type-Curve Method.*—From tables of the function M (u, $\beta$) and $\bar{E}$ of Eq. 40 [M (u, 0.55)], a type curve is prepared on logarithmic paper, and the observed data are plotted on logarithmic paper of the same scale. The two curves are superimposed and adjusted to obtain the matching position shown in Fig. 8. The values $\bar{E}$ [M (u, 0.55)], (1/u), $\bar{s}$, and t at the matching point shown are 0.38, 10, 1 ft, and 1,000 min $(6 \times 10^{-4}$ sec), respectively. From Eq. 40, K is calculated as $4.84 \times 10^{-4}$ fps, and from $S_s = 4\ K\ t/(1/u)\ r^2$, $S_s$ is calculated as $3.5 \times 10^{-4}$ ft$^{-1}$. The value of $(1/u_d)$ at the point of departure is 9. Then, from the relation $b = (1/2)\left[r\beta + r\sqrt{5/u_d}\right]$, b is calculated as 695 ft. The transmissivity of the aquifer is T = Kb = 0.319 sq ft per sec, and the storage coefficient is $S = S_s b = 0.23$.

*Inflection-Point Method.*—The observed semilogarithmic plot is prepared as shown in Fig. 7 (a). The tangent to this curve in the region of the curve inflection is constructed. Its slope $m_i$ is measured as 1.55 ft per cycle. f(x) $= \beta^2/\sqrt{\pi} = (0.55)^2/\sqrt{\pi} = 0.17$. From Table 1, x = 0.371, and erf(x) = 0.40. $u_i = (x/\beta)^2 = 0.454$, and exp (-0.454) = 0.635. From Eq. 17, K is computed as $4.78 \times 10^{-4}$ fps. From tables of the function M (u, $\beta$) [or the type curve of



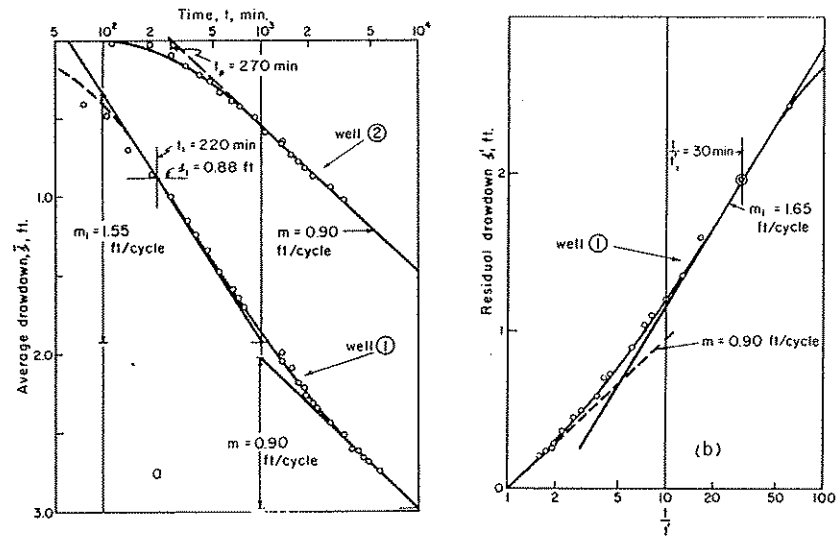


FIG. 7.—a) TIME-DRAWDOWN VARIATION IN THE OBSERVATION WELLS OF THE OLSON TEST; b) LOG (t/t') - RESIDUAL DRAWDOWN VARIATION IN WELL (1) OF THE OLSON TEST

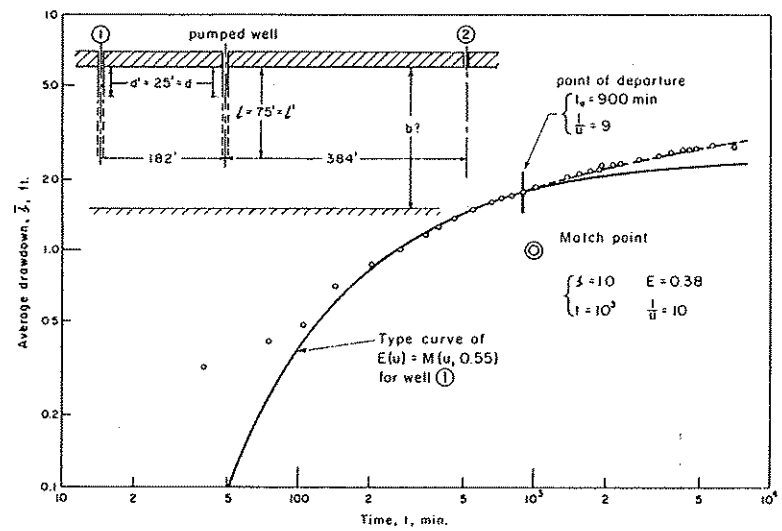


FIG. 8.—OBSERVED DATA AND THE THEORETICAL TYPE CURVE FOR WELL (1) OF THE OLSON TEST

M (u, 0.55)], M $(u_i, 0.55) \approx 0.33$. From Eq. 40, $\bar{s}_i = 0.88$ ft. From Fig. 7 (a), $t_i = 220$ min or 13,200 sec. Hence, $S_s = 4\ K\ t_i\ u_i/r^2 = 3.46 \times 10^{-4}$ ft$^{-1}$. The ultimate straight line of the curve in Fig. 7 (a) is constructed. Its measured slope is 0.90 ft per cycle. From Eq. 21, Kb is calculated as 0.326 ft per sec. Consequently, $b = \left(\frac{Kb}{K}\right) = \left(0.326/4.78 \times 10^{-4}\right) = 680$ ft. The storage coefficient is $S = S_s b = 0.235$.

The calculations presented are those of a second trail. In the first trial, hydraulic conductivity as obtained by using the inflection-point method was underestimated. Apparently, this resulted from drawing "the two tangents" of the observed curve steeper than was necessary. The coefficient obtained by using the type-curve method was overestimated because the best matching position was not obtained, it appears, on the first trial. The two tangents were redrawn with flatter slopes in the first case, and another fitting position was obtained in the second case, because the observed data permitted these adjustments. The calculations were repeated, with the results being those previously presented.

It should be pointed out also that because of a large variation of the well discharge within probably the first 100 min of the pumping period (because of pump characteristics), the observed points within that period were rejected. In addition, one should note that during the very early period of pumping, the observed data generally do not follow the theoretical drawdown equation, because of the probable variation of the formation coefficients, especially if the flow is unconfined.

*Analysis of Drawdown Data from Observation Well (2).*—The observed semilogarithmic time-drawdown curve for this well (see Fig. 7 (a) does not show a curve inflection. The well is known to be of practically zero penetration, and the aquifer is known to be deep, its thickness, as estimated from analysis of data from well (1), is about 700 (actually an average of 670) ft; consequently, $(r/b) = (384/740) < 1.5$. Therefore, the two possibilities for which Theis' or Jacob's method does apply are ruled out, although the general trend of the time-drawdown variation resembles that given by the Theis formula.

*Jacob's Method Adjusted for Partial Penetration.*—Although there are enough points on the curved part of the observed semilogarithmic plot to warrant a fair solution by using the type-curve method (drawdown equation is Eq. 1, with $z = 0$), Jacob's method adjusted for partial penetration is here followed for the purpose of illustrating the procedure, which is as follows:

The slope m of the straight portion of the curve is measured as 0.90 ft per cycle, and $t_p = 270$ min (16,200 sec) [see Fig. 7 (a)]. From Eq. 21, $T = K\ b$ is calculated as 0.326 sq ft per sec. From Eq. 6, $f_s$ is calculated, using a value of $b = 700$ ft (only five terms of the infinite series are needed), as 0.81. Then $\exp(0.81) = 2.25$. From $S = \left[2.25\ T\ t_p \exp\left(f_s\right)/r^2\right]$, the storage coefficient is calculated as 0.18.

*Analysis of Recovery Data from Well (1).*—Fig. 7 (b) is the observed curve of residual drawdown s' versus the logarithm of (t/t'). The tangent in the region of the curve inflection is constructed; its measured slope $m_i = \Delta s'$ per cycle = 1.65 ft per cycle. By inspection, the inflection point is located at $(t/t')_i = 30$.

Because $(t/t')_i < 100$, all terms of Eq. 31 are needed. The computation is as follows:

$$\left(t_o/t'\right) = (t/t')_i - 1 = 29$$



and

$$\left(t_i/t_o\right) = (t/t')_i/\left(t_o/t'_i\right) = (30/29) = 1.035.$$

And

$$f(x) = \beta^2/\sqrt{\pi} = (0.55)^2/\sqrt{\pi} = 0.17$$

(see Eq. 40 for values of $\beta$ and c), for which Table 1 gives $x = 0.371$ and erf(x) = 0.40. Then $u'_i = (x/\beta)^2 = 0.454$; $\exp\left(-u'_i\right) = \exp(-0.454) = 0.635$; $\exp\left(-u'_i/(t/t')_i\right) = \exp(-0.454/30) = 0.985$; and $\mathrm{erf}\left(x/\sqrt{(t/t')_i}\right) = \mathrm{erf}\left(0.371/\sqrt{30}\right) = 0.077$. From Eq. 31, $K = \left[(2.3)(1.6)/(400\pi)(1.65)\right]\left[(1.035)(0.635)\ (0.4) - (0.985)\ (0.077)/(29)\right] = \left(1.77 \times 10^{-3}\right)\ (0.264 - 0.003) = 4.63 \times 10^{-4}$ fps. From $\left(t_o/t'_i\right) = 29$ and $t_o = 6{,}630$ min (period of pumping), $t'_i = 229$ min (13,700 sec). Hence, $S_s = \left(4\ Kt'_i\ u'_i/r^2\right) = 3.46 \times 10^{-4}$ ft$^{-1}$.

The straight portion of the lower left portion of the observed curve is not clearly discernible. However, for the purpose of illustrating the procedure, the straight line is constructed as shown by the dashed line in Fig. 7 (b). Its measured slope m is 0.90 ft per cycle. Then, from Eq. 29 K b is computed as 0.326 sq ft per sec. Finally, $b = (K\ b)/K = \left(0.326/4.63 \times 10^{-4}\right) = 705$ ft, and $S = S_s b = 0.244$.

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## APPENDIX.—NOTATION

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The following symbols, adopted for use in this paper, conform essentially with "American Standard Letter Symbols for Hydraulics" (ASA Z10.2-1942), prepared by a committee of the American Standards Association with Society representation and approved by the Association in 1942:

b = thickness of aquifer; dimension L;

c = constant, depending on the well discharge, length of well screen, and on the hydraulic conductivity of the aquifer; dimension L;

d = depth from the top of the aquifer of the unscreened portion (unperforated section of the casing) of the pumped well, L;

d' = depth from the top of the aquifer of the unscreened portion of the observation well, L;

E = $E\ (u, 1/r, d/r, z/r = M\left[u, (1+z)/r\right] - M\left[u, (d+z)/r\right] + M\left[u, (1-z)/r\right] - M\left[u, (d-z)/r\right]$;

$\bar{E}$ = $\bar{E}\ (u, 1/r, d/r, 1'/r, d'/r) = \left[E \text{ with } z = (1' + d')/2\right]$;

erf(x) = $\frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy$ = The error function;

erfc(x) = 1 - erf(x) = The complement of the error function;

$f$ = $f(u, r/b, l/b, d/b, z/b) = \left[2b/\pi\,(l - d)\right] \sum_{n=1}^{\infty} (1/n) \left[\sin(n\pi\, l/b) - \sin(n\pi\, d/b)\right] \cos(n\pi\, z/b)\, W(u, n\pi\, r/b)$;

$f_s$ = $\left[f, \text{ with } 2\,K_o(n\pi\, r/b) \text{ replacing } W(u, n\pi\, r/b)\right]$;

$\bar{f}$ = $\bar{f}(u, r/b, l/b, d/b, l'/b, d'/b) = \left[2b^2/\pi^2\,(l' - d')\right] \sum_{n=1}^{\infty} (1/n^2) \left[\sin(n\pi\, l/b) - \sin(n\pi\, d/b)\right]\left[\sin(n\pi\, l'/b) - \sin(n\pi\, d'/b)\right] W\left(u, \frac{n\pi\, r}{b}\right)$;

$\bar{f}_s$ = $\left[\bar{f}, \text{ with } 2\,K_o(n\pi\, r/b) \text{ replacing } W(u, n\pi\, r/b)\right]$;

$f_2$ = $f_2(u, b/r, \beta) = \sum_{n=1}^{\infty} M\left[u, \left(\frac{2\,n\,b}{r} + \beta\right)\right] - M\left[u, \left(\frac{2\,n\,b}{r} - \beta\right)\right]$;

$K$ = hydraulic conductivity of the aquifer, $L\,T^{-1}$;

$K_o(x)$ = the zero-order modified Bessel function of the second kind;

$l$ = depth of pentration of the pumped well, L;

$l'$ = depth of penetration of an observation hole, L;

$M(u,\beta)$ = $\int_u^{\infty} \frac{e^{-y}}{y} \operatorname{erf}\left(\beta\sqrt{y}\right) dy$, tabular values of which are available;

$Q$ = constant well discharge, $L^3\,T^{-1}$;

$r$ = radial distance measured from center of well, L;

$r_w$ = effective radius of the pumped well, L;

$S$ = $bS_s$ = storage coefficient;

$S_s$ = specific storage (volume of water from storage by a unit volume of the aquifer under a unit head decline), $L^{-1}$;

$s$ = drawdown of piezometric surface at any time and at any point in the aquifer (drawdown in piezometers), L;

$\bar{s}$ = average drawdown in observation holes, L;

$s'$ = residual drawdown in piezometers, L;

$\bar{s}'$ = residual drawdown in observation holes, L;

$T$ = $Kb$ = transmissivity of the aquifer, $L^2\,T^{-1}$;

$t$ = time since pumping started, T;

$t_o$ = period of pumping, T;



$t'$ = time since pumping stopped, T;

$u$ = $\left(r^2 S_s/4\,K\,t\right)$;

$u'$ = $\left(r^2 S_s/4\,K\,t'\right)$;

$W(u)$ = $\int_u^{\infty} \frac{e^{-y}}{y}\, dy$ = well function of (u) for nonleaky aquifers for which tables are available;

$W(u,x)$ = $\int_u^{\infty} \frac{dy}{y} \exp\left(-y - x^2/4y\right)$ = well function of (u and x) for leaky aquifers for which tables are available;

$z$ = vertical coordinate measured from the top of the aquifer, positive downward; and

$\beta$ = one of the parameters of the function $M(u, \beta)$, also it is a constant that depends on one or all of the variables, $l$, $l'$, $d$, $d'$, $z$, $r$, $b$.