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RESEARCH ARTICLE

Analytical solutions of three-dimensional contaminant transport in uniform flow field in porous media: A library

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Abstract The purpose of this study is to present a library of analytical solutions for the three-dimensional contaminant transport in uniform flow field in porous media with the first-order decay, linear sorption, and zero-order production. The library is constructed using Green's function method (GFM) in combination with available solutions. The library covers a wide range of solutions for various conditions. The aquifer can be vertically finite, semi-infinitive or infinitive, and laterally semi-infinitive or infinitive. The geometry of the sources can be of point, line, plane or volumetric body; and the source release can be continuous, instantaneous, or by following a given function over time. Dimensionless forms of the solutions are also proposed. A computer code FlowCAS is developed to calculate the solutions. Calculated results demonstrate the correctness of the presented solutions. The library is widely applicable to solve contaminant transport problems of one- or multiple- dimensions in uniform flow fields.

Keywords solution library, contaminant transport, analytical solution, dispersion and advection, porous media, type curve, Green's function method (GFM)

1 Introduction

Analytical solutions of contaminant transport in porous media can be applied in many fields. They can provide better physical insights into the problems; thus, are the basic tools to reveal the mechanism of contaminant transport phenomena. Analytical solutions can help us to analyze the influence of parameters and boundary conditions, measure the field parameters related to contaminant transport, and verify the numerical models, as they are free of numerical dispersions attached to numerical simulations, especially in the advection dominated transport [1]. They can also be used to predict the movement of contaminant plumes and design the experiments in the field or laboratory scales [2].

Analytical solutions of contaminant transport have been widely studied in the last three decades. Many solutions have been published before and new ones are still appearing. For instance, Cleary and Adrian [3], Sauty and Pierre [4], and van Genuchten [5] have provided analytical solutions for one-dimensional transport with the first-type (Dirichlet) and third-type (Cauchy) boundary conditions, respectively. Hunt [6], Latinopoulos et al [7], and Wilson and Miller [8] have investigated the two dimensional transport models for the continuous release of a point source. Batu has studied the two-dimensional analytical solutions for the plane sources with the first- and the second-type boundary conditions [9,10]. Quezada and Clement developed a generalized solution to multi-dimensional multi-species transport equations coupled through sequential first-order reactions [11]. Recently, Srinivasan and Clement derived a closed-form solution to the problem [12,13].

For the study of three-dimensional contaminant transport in uniform flow field, one of the most widely used analytical solutions is the approximate one proposed by Domenico [14,15]. The solution is expressed in a simple way without integration. Neville has obtained the solution for the vertical rectangle source by taking the source as the boundary condition of the first type [16]. Leij et al. have studied the analytical solutions for non-equilibrium solute transport in three-dimensional porous media using the Laplace, Fourier, and Hankel transforms [17,18]. Later in 2000, Leij et al. have provided a useful Green's function table using the Green's function method (GFM) for various plane sources that can be used to construct solutions [19]. They also illustrated three solutions for a semi-infinitive aquifer. Park and Zhan obtained a series of analytical solutions of contaminant transport from finite one-, two-, and three- dimensional sources in a finite-thickness aquifer by GFM [20]. In addition, many other authors have also published some solutions for different sources and boundary conditions [21–26].

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The purpose of this paper is to present a library of analytical solutions of contaminant transport in uniform flow field in porous media. The users can either use the solutions presented in the paper or construct their own solutions by using the directional solutions and their integration results over directional source lengths to solve their specific problems. The types of contaminant sources can be of point, line, plane or volumetric body; their release duration can be instantaneous, temporary or continuous and their strengths or concentrations can be constant or variable over time. The first-order decay or attenuation and linear retardation process (Henry module) are considered. The aquifer can be vertically infinitive, semi-infinitive or finite and horizontally infinitive or semi-infinitive. By using the method of imaging and superposition principles, the problems of linear boundaries, multiple sources, sources of irregular shapes, and random change of source strengths and concentrations can also be solved.

2 Mathematical model and solution methods

2.1 Assumptions

The assumptions embedded in the analytical solutions are actually the constraint conditions for their application. Therefore, it is important to know the conditions and limitations before we make use of the solutions for practical purpose. The following conditions are assumed in this study:

Studied domain The aquifer or the studied domain is vertically finite with constant thickness or semi-finite, horizontally infinite or semi-finite in x- or y-directions, homogeneous, and anisotropic.

Source type There exist sources of point, line, plane or volumetric body, either regular or irregular in shape. They can be treated either as a source term inside the domain or as a boundary condition if the source concentration is provided. The corresponding domain for the former case is horizontally infinite, while for the latter case, is horizontally semi-finite. The source strength can be constant or time-dependent but not space- independent.

Flow field Fluid flow is steady and uniform along the x- or xz-directions.

Retardation Partitioning between dissolved and sorbed phases is instantaneous and reversible, and can be represented by the linear sorption process.

Attenuation Solute undergoes first-order degradation or decay. The rate constants can be the same or different in the dissolved and sorbed phases, or zero in the sorbed phase.

Initial condition There is no target contaminant in the studied domain at the initial time.

Transport The contaminant transport in the domain is assumed to be three-dimensional. The components of hydrodynamic dispersion coefficient tensor are known with its principal directions parallel to the x-, y-, and z-axis respectively.

It should be noted that from the view of the constraints applied for the analytical solutions, the above assumptions are relatively widely applicable, compared to more strict constraints embedded in other known analytical solutions.

2.2 Mathematical model

The three-dimensional advection- dispersion equation with first-order decay and zero order source or production under the above assumptions can be written as [2]

$$L(C) = I/n, \tag{1}$$

where the linear operator L is defined by

$$L(C) = \frac{\partial C}{\partial t} - D_x \frac{\partial^2 C}{\partial x^2} - D_y \frac{\partial^2 C}{\partial y^2} - D_z \frac{\partial^2 C}{\partial z^2} + u\frac{\partial C}{\partial x} + \lambda C, \tag{2}$$

and subject to initial condition

$$C(x,y,z,t)|_{t=0} = 0, x,y,z \in D, \tag{3}$$

and boundary conditions of various types, which will be discussed later, where C(x, y, z, t) [$\mathrm{ML^{-3}}$] is the solute concentration at location (x, y, z) at time t; D_x, D_y, and D_z [$\mathrm{L^2T^{-1}}$] are principal dispersion coefficients; λ [$\mathrm{T^{-1}}$] is the first-order constant for decay in liquid and sorbed phases; u [$\mathrm{LT^{-1}}$] is the pore- water velocity along the x-axis; b [L] and n are the thickness and the porosity of the aquifer, respectively; D is the studied domain; and I [$\mathrm{ML^{-3}T^{-1}}$] is the zero-order source term.

Note that I exists only in the space occupied by the source, while for the rest of the domain, I is equal to zero. For instance, if source occupies a space of parallelepiped body, $x \in [0, x_0]$, $y \in [-y_0, y_0]$, and $z \in [z_0, z_1]$,

$$I = \begin{cases} I_0 f(t), 0 < x < x_0,\ -y_0 < y < y_0,\ z_0 < z < z_1,\ t > 0, \\ 0, \text{ otherwise}, \end{cases} \tag{4}$$

where I_0 [$\mathrm{ML^{-3}T^{-1}}$] is the initial volumetric mass released by the source per unit time; $f(t)$ is a dimensionless known function, representing the changes of source strength with time.

The boundary conditions are set as follows:

1) **Horizontal infinitive boundaries** If the source term is inside the domain and source strength is provided, the horizontal domain is set to infinitive and the concentration in infinitive boundaries is set to zero, that is,

$$C(x,y,z,t)|_{x\to\pm\infty} = 0,\ t > 0; \tag{5}$$

$$C(x,y,z,t)|_{y\to\pm\infty} = 0,\ t > 0. \tag{6}$$

2) **Horizontal semi-infinitive boundaries** If the source concentration is provided, it can be treated as a boundary condition of the first type. The function of sources is implemented by the boundary condition while I in Equation (1) is set to zero. For instance, if a vertical rectangle source is along the yz- plane at $x = 0$, the studied domain can be set to semi- infinitive in the x-direction, $x \in [0, +\infty)$, and the boundary condition at $x = 0$ is expressed as

$$C(x,y,z,t)|_{x=0} = \begin{cases} C_0,\ y,\ z\in\text{source},\ t > 0, \\ 0,\ \text{otherwise}. \end{cases} \tag{7}$$

3) **Vertical infinitive boundaries** If an aquifer can be considered as infinitive in thickness, the vertical domain in the z-direction is infinitive, and the vertical infinitive boundary is set to the first type as

$$C(x,y,z,t)|_{z\to\pm\infty} = 0,\ t > 0. \tag{8}$$

4) **Vertical semi-infinitive boundaries with the source concentration provided on top boundaries** The vertical domain is $z \in [0, +\infty)$, and the top boundary condition is

$$C(x,y,z,t)|_{z=0} = \begin{cases} C_0,\ x,\ y\in\text{source},\ t > 0, \\ 0,\ \text{otherwise}. \end{cases} \tag{9}$$

5) **Vertical finite boundaries** If an aquifer is of constant thickness, b, the vertical domain is $z \in [0, b]$. It is assumed that the contaminant can not penetrate through the top and bottom boundaries, that is,

$$\left.\frac{\partial C}{\partial z}\right|_{z=0} = \left.\frac{\partial C}{\partial z}\right|_{z=b} = 0,\quad -\infty < x,y < \infty,\ t > 0. \tag{10}$$

2.3 Solution method

Solutions to Eq. (1) exist for a variety of boundary conditions as stated above. The solutions can be derived by making use of traditional integration transform methods, such as Laplace method, or more recent Green's function method (GFM). The GFM can be briefly described in the following steps [19,20,27,28]:

1) Deriving the solutions for an instantaneous point source in an infinitive, semi-finite and finite aquifer. The governing equation for an infinite aquifer is

$$L(C) = \frac{M_p}{n}\delta(x-x',y-y',z-z',t-\tau), \tag{11}$$

and the corresponding differential equation of the Green's function is [19,29]

$$L^*(G) = \delta(x-x',y-y',z-z',t-\tau), \tag{12}$$

where L^* is the adjoint differential operator:

$$L^*(G) = -\frac{\partial G}{\partial \tau} - D_x\frac{\partial^2 G}{\partial x'^2} - u\frac{\partial G}{\partial x'} - D_y\frac{\partial^2 G}{\partial y'^2} - D_z\frac{\partial^2 G}{\partial z'^2} + \lambda G, \tag{13}$$

where G is the Green's function; M_p [M] is the mass released instantaneously at point (x', y', z') at time τ; δ [$L^{-3}T^{-1}$] is the Dirac delta function.

The GFM indicates that a three-dimensional solution is the product of the three directional solutions [19,27]. The directional solutions for directions x, y and z with different boundary conditions are summarized in Table 1. For instance, the directional solution in the x-direction for an instantaneous point source is shown in Eq. (14).

Note that if an aquifer is vertically semi-finite or finite with a constant thickness b, the contaminant cannot penetrate through its top or bottom boundaries. To solve the problem, the method of imaging is applied [30]. Take the finite thick aquifer as an example, the source and the top and bottom boundaries are imaged for infinite numbers of times in the z-direction. Therefore, the boundary value problem can be represented by a vertically infinitive problem with infinite numbers of the image sources and the original source [20,2], as shown in Eq. (20).

From Table 1 it is easy to construct solutions for the instantaneous point sources. For instance, the analytical solutions for the infinitively and finitely thick aquifers are [20,29]

$$C(x,y,z,t) = \frac{M_p}{8n\sqrt{\pi^3 D_x D_y D_z (t-\tau)^3}}\exp[-\lambda(t-\tau)] \times\exp\left\{-\frac{[(x-x')-u(t-\tau)]^2}{4D_x(t-\tau)}\right\} \times\exp\left[-\frac{(y-y')^2}{4D_y(t-\tau)}\right]\exp\left[-\frac{(z-z')^2}{4D_z(t-\tau)}\right], \tag{22}$$

and

$$C(x,y,z,t) = \frac{M_p}{4\pi nb(t-\tau)\sqrt{D_x D_y}}\exp[-\lambda(t-\tau)] \times\exp\left\{-\frac{[(x-x')-u(t-\tau)]^2}{4D_x(t-\tau)}\right\} \times\exp\left[-\frac{(y-y')^2}{4D_y(t-\tau)}\right]\left[1+2\sum_{n=1}^{\infty}\cos\frac{n\pi(z_0-z')}{b} \times\cos\frac{n\pi(z-z')}{b}\exp\left(-\frac{D_z n^2\pi^2}{b^2}(t-\tau)\right)\right], \tag{23}$$

Table 1 Directional solutions for the instantaneous point sources

direction	condition	directional solution	
x	$C\|_{x\to\pm\infty}=0$ $-\infty<x<+\infty$	$G_x=\frac{1}{2\sqrt{\pi D_x(t-\tau)}}\exp\left\{-\frac{[(x-x')-u(t-\tau)]^2}{4D_x(t-\tau)}\right\}$	(14)
	$C\|_{x=0}=\begin{cases}C_0,(y,z)\in\text{source}\\0,\text{otherwise}\end{cases}$ $0<x<+\infty$	$G_x=\frac{x-x'}{2\sqrt{\pi D_x(t-\tau)}}\exp\left\{-\frac{[x-x'-u(t-\tau)]^2}{4D_x(t-\tau)}\right\}$	(15)
y	$C\|_{y\to\pm\infty}=0$ $-\infty<y<+\infty$	$G_y=\frac{1}{2\sqrt{\pi D_y(t-\tau)}}\exp\left[-\frac{(y-y')^2}{4D_y(t-\tau)}\right]$	(16)
	$C\|_{y=0}=\begin{cases}C_0,(x,z)\in\text{source}\\0,\text{otherwise}\end{cases}$ $0<y<+\infty$	$G_y=\frac{y-y'}{2\sqrt{\pi D_y(t-\tau)}}\exp\left[-\frac{(y-y')^2}{4D_y(t-\tau)}\right]$	(17)
z	$C\|_{z\to\pm\infty}=0$ $-\infty<z<+\infty$	$G_z=\frac{1}{2\sqrt{\pi D_z(t-\tau)}}\exp\left[-\frac{(z-z')^2}{4D_z(t-\tau)}\right]$	(18)
	$\partial C/\partial z\|_{z=0}=0$ $C\|_{z\to+\infty}=0$ $0\leqslant z<+\infty$	$G_z=\frac{1}{2\sqrt{\pi D_z(t-\tau)}}\left\{\exp\left[-\frac{(z-z')^2}{4D_z(t-\tau)}\right]+\exp\left[-\frac{(z+z')^2}{4D_z(t-\tau)}\right]\right\}$	(19)
	$\partial C/\partial z\|_{z=0}=0$ $\partial C/\partial z\|_{z=b}=0$ $0\leqslant z\leqslant b$	$G_z=\frac{1}{b}\left\{1+2\sum_{m=1}^{\infty}\cos\frac{m\pi(z_0-z')}{b}\cos\frac{m\pi(z-z')}{b}\exp\left[-\frac{D_z n^2\pi^2}{b^2}(t-\tau)\right]\right\}$	(20)
	$C\|_{z=0}=\begin{cases}C_0,(x,y)\in\text{source}\\0,\text{otherwise}\end{cases}$ $0<z<+\infty$	$G_z=\frac{z-z'}{2\sqrt{\pi D_z(t-\tau)}}\exp\left[-\frac{(x-x')^2}{4D_z(t-\tau)}\right]$	(21)

respectively. When $\tau = 0$, Eqs. (22) and (23) becomes the solutions for the point source of initially instantaneous release [20,2].

2) Obtaining the instantaneous solutions by an analytic integration over the length, area or volume of the source. Taking the above mentioned parallelepiped body source as an example, the source body is located at $x \in [0, x_0]$, $y \in [-y_0, y_0]$, and $z \in [z_0, z_1]$. For an infinitively wide and infinitively thick aquifer,

$$C(x,y,z,t)=\frac{M_v}{8n\sqrt{\pi^3 D_x D_y D_z(t-\tau)^3}}\exp[-\lambda(t-\tau)]\times\int_0^{x_0}\int_{-y_0}^{y_0}\int_{z_0}^{z_1}\exp\left\{-\frac{[(x-x')-u(t-\tau)]^2}{4D_x(t-\tau)}\right\}\times\exp\left[-\frac{(y-y')^2}{4D_y(t-\tau)}\right]\exp\left[-\frac{(z-z')^2}{4D_z(t-\tau)}\right]\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'. \tag{24}$$

The results of the analytic integration over the source length in the corresponding direction, which can be used to construct analytical solutions for various conditions, are summarized in Table 2 for the infinitive, semi-infinitive, and finite boundaries.

From Eq. (24) referred to Table 2, we can easily obtain the analytical solution of the instantaneous release from a volumetric source:

$$C(x,y,z,t)=\frac{M_v}{8n}\exp[-\lambda(t-\tau)]\left[\operatorname{erfc}\frac{x-u(t-\tau)-x_0}{2\sqrt{D_x(t-\tau)}}-\operatorname{erfc}\frac{x-u(t-\tau)}{2\sqrt{D_x(t-\tau)}}\right]\times\left[\operatorname{erfc}\frac{y-y_0}{2\sqrt{D_y(t-\tau)}}-\operatorname{erfc}\frac{y+y_0}{2\sqrt{D_y(t-\tau)}}\right]\times\left[\operatorname{erfc}\frac{z-z_0}{2\sqrt{D_z(t-\tau)}}-\operatorname{erfc}\frac{z+z_0}{2\sqrt{D_z(t-\tau)}}\right], \tag{30}$$

where M_v [$\mathrm{ML^{-3}}$] is the mass released instantaneously at time τ per unit source volume. Similarly, we can obtain the solutions for a finitely thick aquifer. If $\tau = 0$, Eq. (30) becomes the solution for initially instantaneous release.

3) Obtaining the solutions of continuous source release by analytic integration of instantaneous release solutions over time. For a continuous source, integration of Eq. (30)

Table 2 Results of the integration of directional solutions over the source lengths

source range	condition	integration result	
$0-x_0$	$C\|_{x\to\pm\infty}=0$ $-\infty<x<+\infty$	$\frac{1}{2}\left[\mathrm{erfc}\frac{x-u(t-\tau)-x_0}{2\sqrt{D_x(t-\tau)}}-\mathrm{erfc}\frac{x-u(t-\tau)}{2\sqrt{D_x(t-\tau)}}\right]$	(25)
$-y_0-y_0$	$C\|_{y\to\pm\infty}=0$ $-\infty<y<+\infty$	$\frac{1}{2}\left[\mathrm{erfc}\frac{y-y_0}{2\sqrt{D_y(t-\tau)}}-\mathrm{erfc}\frac{y+y_0}{2\sqrt{D_y(t-\tau)}}\right]$	(26)
z_0-z_1	$C\|_{z\to\pm\infty}=0$ $-\infty<z<+\infty$	$\frac{1}{2}\left[\mathrm{erfc}\frac{z-z_1}{2\sqrt{D_z(t-\tau)}}-\mathrm{erfc}\frac{z-z_0}{2\sqrt{D_z(t-\tau)}}\right]$	(27)
z_0-z_1	$\partial C/\partial z\|_{z=0}=0$ $C\|_{z\to+\infty}=0$ $0\leqslant z<+\infty$	$\frac{1}{2}\left[\mathrm{erfc}\frac{z-z_1}{2\sqrt{D_z\tau}}-\mathrm{erfc}\frac{z-z_0}{2\sqrt{D_z\tau}}+\mathrm{erfc}\frac{z+z_0}{2\sqrt{D_z\tau}}-\mathrm{erfc}\frac{z+z_1}{2\sqrt{D_z\tau}}\right]$	(28)
z_0-z_1	$\partial C/\partial z\|_{z=0}=0$ $\partial C/\partial z\|_{z=b}=0$ $0\leqslant z\leqslant b$	$\frac{1}{b}\left[z_1-z_0+\frac{2b}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi z_1}{b}-\sin\frac{n\pi z_0}{b}\right)\cos\frac{n\pi z}{b}\exp\left(-\frac{D_z n^2\pi^2}{b^2}(t-\tau)\right)\right]$	(29)

over time from 0 to t yields

$$C(x,y,z,t)=\frac{I_0}{8n}\int_0^t f(t-\tau)\exp(-\lambda\tau)\times\left[\mathrm{erfc}\frac{x-u\tau-x_0}{2\sqrt{D_x\tau}}-\mathrm{erfc}\frac{x-u\tau}{2\sqrt{D_x\tau}}\right]\times\left[\mathrm{erfc}\frac{y-y_0}{2\sqrt{D_y\tau}}-\mathrm{erfc}\frac{y+y_0}{2\sqrt{D_y\tau}}\right]\times\left[\mathrm{erfc}\frac{z-z_1}{2\sqrt{D_z\tau}}-\mathrm{erfc}\frac{z-z_0}{2\sqrt{D_z\tau}}\right]\mathrm{d}\tau. \quad (31)$$

3 Solution library

By using the above stated GFM in combinations with superposition principle and imaging method, a library of analytical solutions is constructed for a three-dimensional contaminant transport in a uniform flow field.

3.1 Solutions for continuous release

Solutions for the regular continuous source release are summarized in Table 3, with the source types of point, line, plane, and parallel-piped body in a finite and semi-finite aquifer. Some solutions in the Table have been published by other authors as noted, and some are, for the first time, constructed by using the equations in Tables 1 and 2. For the practical application of the solutions, users must be aware of the conditions and constraints embedded with them.

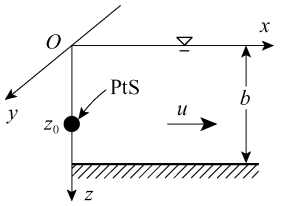
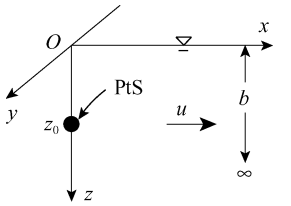
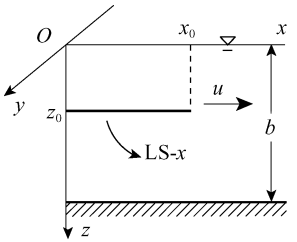
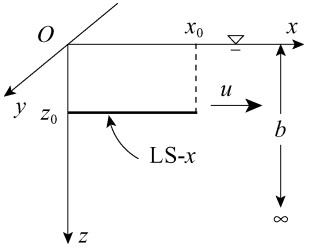
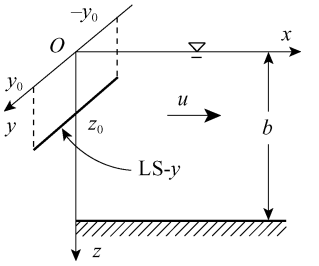
Equations in Table 3 can be easily simplified to obtain the solutions for one- or two-dimensional transport problems. For instance, by letting z_0 equal to 0 and z_1 to b in Eq. (38), the solution of two-dimensional transport is obtained for a point source in horizontal plane view, which is identical to the solution proposed by Wilson and Miller [8].

Most analytical solutions shown in Table 3 include the temporal integrations. In general, the integrations are often evaluated numerically. In this paper, the Gaussian Quadrature method is utilized to numerically evaluate integrations. A computer code named Flow-CAS (Flow and Contaminant transport Analytical Solutions) is created by using Borland Delphi 7. The code can make calculations of all solutions presented in this paper.

3.2 Solutions for instantaneous release

As mentioned in the above section, solutions for the instantaneous release are the basis for deriving the solutions for continuous release. Therefore, the solutions in Table 3 can be easily modified to yield the solutions for instantaneous release. For the cases of given source strengths, by substituting M_p, M_l, M_a, and M_v for I_p, I_l, I_a, and I, respectively, taking out of the temporal integration, and substituting t for τ, the corresponding equations in the Table then become the solutions for the instantaneous release, where M_l [$\mathrm{ML^{-1}}$] and M_a [$\mathrm{ML^{-2}}$] are the mass released per unit source length and per unit source area, respectively. For instance, the solution for the instantaneous release of a point source in a finite aquifer is, according to Eq. (32):

Table 3 Three dimensional analytical solutions for the continuous release of point, line, plane and parallelepiped body sources in uniform flow field

source type/boundary condition	solution	schematic diagram
point source at (0, 0, z_0). (I_p [MT^{-1}] is the contaminant mass released per unit time.)[a]	$C(x,y,z,t)=\frac{1}{4\pi nb\sqrt{D_xD_y}}\int_0^t I_p(t-\tau)\exp\left[-\lambda\tau-\frac{(x-u\tau)^2}{4D_x\tau}-\frac{y^2}{4D_y\tau}\right]$ $\times\left[1+2\sum_{m=1}^{\infty}\cos\frac{m\pi z_0}{b}\cos\frac{m\pi z}{b}\exp\left(-\frac{D_zm^2\pi^2}{b^2}\tau\right)\right]\frac{d\tau}{\tau}$ $-\infty<x,y<\infty, 0\leqslant z\leqslant b, t>0$ (32)	
point source at (0, 0, z_0) (I_p [MT^{-1}] is the contaminant mass released per unit time, in the semi- infinitive thickness aquifer.)	$C(x,y,z,t)=\frac{1}{8n(\pi)^{3/2}\sqrt{D_xD_yD_z}}\int_0^t I_p(t-\tau)\exp\left[-\lambda\tau-\frac{(x-u\tau)^2}{4D_x\tau}-\frac{y^2}{4D_y\tau}\right]$ $\times\left\{\exp\left[-\frac{(z-z_0)^2}{4D_z\tau}\right]+\exp\left[-\frac{(z+z_0)^2}{4D_z\tau}\right]\right\}\frac{d\tau}{\tau^{3/2}}$ $-\infty<x,y<\infty, 0\leqslant z<\infty, t>0$ (33)	
line source in the x-direction from (0, 0, z_0) to (x_0, 0, z_0) (I_l [$ML^{-1}T^{-1}$] is the contaminant mass released per unit source length per unit time.)[a]	$C(x,y,z,t)=\frac{1}{4nb\sqrt{\pi D_y}}\int_0^t I_l(t-\tau)\exp\left(-\lambda\tau-\frac{y^2}{4D_y\tau}\right)\left[\mathrm{erfc}\frac{x-u\tau-x_0}{2\sqrt{D_x\tau}}-\mathrm{erfc}\frac{x-u\tau}{2\sqrt{D_x\tau}}\right]$ $\times\left[1+2\sum_{m=1}^{\infty}\cos\frac{m\pi z_0}{b}\cos\frac{m\pi z}{b}\exp\left(-\frac{D_zm^2\pi^2}{b^2}\tau\right)\right]\frac{d\tau}{\sqrt{\tau}}$ $-\infty<x,y<\infty, 0\leqslant z\leqslant b, t>0$ (34)	
line source in the x-direction from (0, 0, z_0) to (x_0, 0, z_0) (I_l [$ML^{-1}T^{-1}$] is the contaminant mass released per unit source length per unit time, in the semi-infinitive thickness aquifer.)	$C(x,y,z,t)=\frac{1}{8n\pi\sqrt{D_yD_z}}\int_0^t I_l(t-\tau)\exp\left[-\lambda\tau-\frac{y^2}{4D_y\tau}\right]\times\left\{\exp\left[-\frac{(z-z_0)^2}{4D_z\tau}\right]\right.$ $\left.+\exp\left[-\frac{(z+z_0)^2}{4D_z\tau}\right]\right\}\left[\mathrm{erfc}\frac{x-u\tau-x_0}{2\sqrt{D_x\tau}}-\mathrm{erfc}\frac{x-u\tau}{2\sqrt{D_x\tau}}\right]\frac{d\tau}{\tau}$ $-\infty<x,y<\infty, 0\leqslant z<\infty, t>0$ (35)	
line source in the y-direction from (0, $-y_0$, z_0) to (0, y_0, z_0) (I_l [$ML^{-1}T^{-1}$] is the contaminant mass released per unit source length per unit time.)[a]	$C(x,y,z,t)=\frac{1}{4nb\sqrt{\pi D_x}}\int_0^t I_l(t-\tau)\exp\left[-\lambda\tau-\frac{(x-u\tau)^2}{4D_x\tau}\right]\left[\mathrm{erfc}\frac{y-y_0}{2\sqrt{D_y\tau}}-\mathrm{erfc}\frac{y+y_0}{2\sqrt{D_y\tau}}\right]$ $\times\left[1+2\sum_{m=1}^{\infty}\cos\frac{m\pi z_0}{b}\cos\frac{m\pi z}{b}\exp\left(-\frac{D_zm^2\pi^2}{b^2}\tau\right)\right]\frac{d\tau}{\sqrt{\tau}}$ $-\infty<x,y<\infty, 0\leqslant z\leqslant b, t>0$ (36)	

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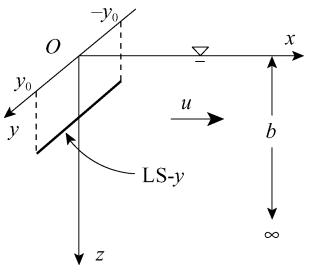
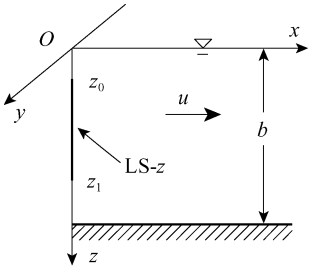
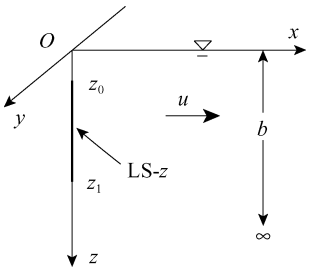
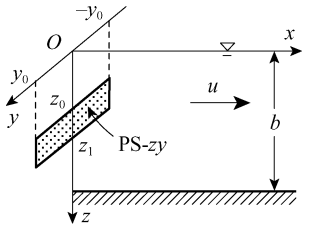
source type/boundary condition	solution	schematic diagram
line source in the y-direction from $(0, -y_0, z_0)$ to $(0, y_0, z_0)$ (I_l [$\mathrm{ML^{-1}T^{-1}}$] is the contaminant mass released per unit source length per unit time, in the semi-infinitive thickness aquifer.)	$C(x,y,z,t)=\frac{1}{8n\pi\sqrt{D_xD_z}}\int_0^t I_l(t-\tau)\exp\left[-\lambda\tau-\frac{(x-u\tau)^2}{4D_x\tau}\right]\times\left\{\exp\left[-\frac{(z-z_0)^2}{4D_z\tau}\right]+\exp\left[-\frac{(z+z_0)^2}{4D_z\tau}\right]\right\}\left[\mathrm{erfc}\frac{y-y_0}{2\sqrt{D_y\tau}}-\mathrm{erfc}\frac{y+y_0}{2\sqrt{D_y\tau}}\right]\frac{\mathrm{d}\tau}{\tau}$ $-\infty<x,y<\infty, 0\leqslant z<\infty, t>0$ (37)	
line source in the z-direction from $(0, 0, z_0)$ to $(0, 0, z_1)$ (I_l [$\mathrm{ML^{-1}T^{-1}}$] is the contaminant mass released per unit source length per unit time.)[a]	$C(x,y,z,t)=\frac{1}{4\pi nb\sqrt{D_xD_y}}\int_0^t I_l(t-\tau)\exp\left[-\lambda\tau-\frac{(x-u\tau)^2}{4D_x\tau}-\frac{y^2}{4D_y\tau}\right]\times\left[z_1-z_0+\frac{2b}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left(\sin\frac{m\pi z_1}{b}-\sin\frac{m\pi z_0}{b}\right)\cos\frac{m\pi z}{b}\exp\left(-\frac{D_zm^2\pi^2}{b^2}\tau\right)\right]\frac{\mathrm{d}\tau}{\tau}$ $-\infty<x,y<\infty, 0\leqslant z\leqslant b, t>0$ (38)	
line source in the z-direction from $(0, 0, z_0)$ to $(0, 0, z_1)$ (I_l [$\mathrm{ML^{-1}T^{-1}}$] is the contaminant mass released per unit source length per unit time,in the semi-infinitive thickness aquifer.)	$C(x,y,z,t)=\frac{1}{8n\pi\sqrt{D_xD_y}}\int_0^t I_l(t-\tau)\exp\left[-\lambda\tau-\frac{(x-u\tau)^2}{4D_x\tau}-\frac{y^2}{4D_y\tau}\right]\times\left[\mathrm{erfc}\frac{z-z_1}{2\sqrt{D_z\tau}}-\mathrm{erfc}\frac{z-z_0}{2\sqrt{D_z\tau}}+\mathrm{erfc}\frac{z+z_0}{2\sqrt{D_z\tau}}-\mathrm{erfc}\frac{z+z_1}{2\sqrt{D_z\tau}}\right]\frac{\mathrm{d}\tau}{\tau}$ $-\infty<x,y<\infty, 0\leqslant z<\infty, t>0$ (39)	
vertical rectangle source in the yz plane at $x = 0$, $y \in [-y_0, y_0]$, $z \in [z_0, z_1]$ (I_a [$\mathrm{ML^{-2}T^{-1}}$] is the contaminant mass released per unit source area per unit time.)[a]	$C(x,y,z,t)=\frac{1}{4nb\sqrt{\pi D_x}}\int_0^t I_a(t-\tau)\exp\left[-\lambda\tau-\frac{(x-u\tau)^2}{4D_x\tau}\right]\left[\mathrm{erfc}\frac{y-y_0}{2\sqrt{D_y\tau}}-\mathrm{erfc}\frac{y+y_0}{2\sqrt{D_y\tau}}\right]\times\left[z_1-z_0+\frac{2b}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left(\sin\frac{m\pi z_1}{b}-\sin\frac{m\pi z_0}{b}\right)\cos\frac{m\pi z}{b}\exp\left(-\frac{D_zm^2\pi^2}{b^2}\tau\right)\right]\frac{\mathrm{d}\tau}{\sqrt{\tau}}$ $-\infty<x,y<\infty, 0\leqslant z\leqslant b, t>0$ (40)	

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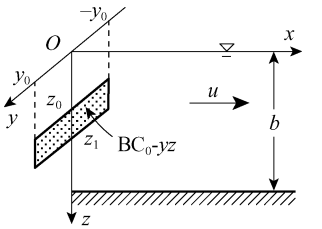
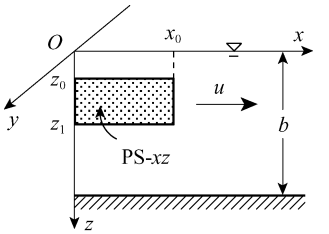
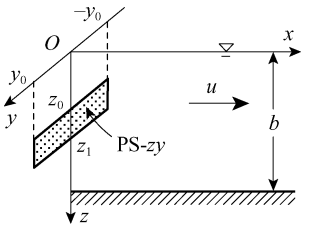
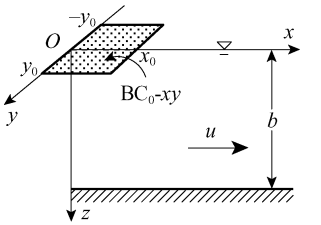
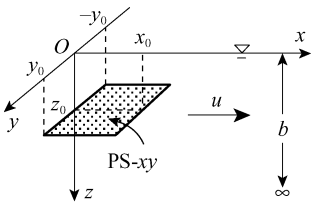
source type/boundary condition	solution	schematic diagram
vertical boundary in the yz plane at $x = 0$ (The rectangle source is in $y \in [-y_0, y_0]$, $z \in [z_0, z_1]$; $C\|_{x=0} = \begin{cases} C_0, \text{when}(y,z) \in \text{source} \\ 0, \text{otherwise} \end{cases}$ where C_0 [ML^{-3}] is the source concentration.)[b]	$C(x,y,z,t) = \frac{C_0 x}{4b\sqrt{\pi D_x}} \int_0^t \exp\left[-\lambda\tau - \frac{(x-u\tau)^2}{4D_x\tau}\right] \left[\text{erfc}\frac{y-y_0}{2\sqrt{D_y\tau}} - \text{erfc}\frac{y+y_0}{2\sqrt{D_y\tau}}\right]$ $\times \left[z_1 - z_0 + \frac{2b}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left(\sin\frac{m\pi z_1}{b} - \sin\frac{m\pi z_0}{b}\right)\cos\frac{m\pi z}{b}\exp\left(-\frac{D_z m^2\pi^2}{b^2}\tau\right)\right]\frac{d\tau}{\tau^{3/2}}$ $x > 0, -\infty < y < \infty, 0 \leqslant z \leqslant b, t > 0$ (41)	
vertical rectangle source in the xz plane at $y = 0$, $x \in [0, x_0]$, $z \in [z_0, z_1]$ (I_a [$ML^{-2}T^{-1}$] is the contaminant mass released per unit source area per unit time.)[a]	$C(x,y,z,t) = \frac{1}{4nb\sqrt{\pi D_y}} \int_0^t I_a(t-\tau)\exp\left(-\lambda\tau - \frac{y^2}{4D_y\tau}\right)\left[\text{erfc}\frac{x-u\tau-x_0}{2\sqrt{D_x\tau}} - \text{erfc}\frac{x-u\tau}{2\sqrt{D_x\tau}}\right]$ $\times \left[z_1 - z_0 + \frac{2b}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left(\sin\frac{m\pi z_1}{b} - \sin\frac{m\pi z_0}{b}\right)\cos\frac{m\pi z}{b}\exp\left(-\frac{D_z m^2\pi^2}{b^2}\tau\right)\right]\frac{d\tau}{\sqrt{\tau}}$ $-\infty < x,y < \infty, 0 \leqslant z \leqslant b, t > 0$ (42)	
horizontal rectangle source in the xy plane at $z = z_0$, $x \in [0, x_0]$, $y \in [-y_0, y_0]$ (I_a [$ML^{-2}T^{-1}$] is the contaminant mass released per source area per unit time.)[a]	$C(x,y,z,t) = \frac{1}{4nb}\int_0^t I_a(t-\tau)\exp(-\lambda\tau)\left[\text{erfc}\frac{x-u\tau-x_0}{2\sqrt{D_x\tau}} - \text{erfc}\frac{x-u\tau}{2\sqrt{D_x\tau}}\right]$ $\times \left[\text{erfc}\frac{y-y_0}{2\sqrt{D_y\tau}} - \text{erfc}\frac{y+y_0}{2\sqrt{D_y\tau}}\right]\left[1 + 2\sum_{m=1}^{\infty}\cos\frac{m\pi z_0}{b}\cos\frac{m\pi z}{b}\exp\left(-\frac{D_z m^2\pi^2}{b^2}\tau\right)\right]d\tau$ $-\infty < x,y < \infty, 0 \leqslant z \leqslant b, t > 0$ (43)	
horizontal boundary in the xy plane at $z = 0$ (The rectangle source is in $x \in [0, x_0]$, $y \in [-y_0, y_0]$; $C\|_{z=0} = \begin{cases} C_0, \text{when}(x,y) \in \text{source} \\ 0, \text{otherwise} \end{cases}$ where C_0 [ML^{-3}] is the source concentration.)	$C(x,y,z,t) = \frac{C_0 z}{4b}\int_0^t \exp(-\lambda\tau)\left[\text{erfc}\frac{x-u\tau-x_0}{2\sqrt{D_x\tau}} - \text{erfc}\frac{x-u\tau}{2\sqrt{D_x\tau}}\right]$ $\times \left[\text{erfc}\frac{y-y_0}{2\sqrt{D_y\tau}} - \text{erfc}\frac{y+y_0}{2\sqrt{D_y\tau}}\right]\left[1 + 2\sum_{m=1}^{\infty}\cos\frac{m\pi z}{b}\exp\left(-\frac{D_z m^2\pi^2}{b^2}\tau\right)\right]\frac{d\tau}{\tau}$ $-\infty < x,y < \infty, 0 < z \leqslant b, t > 0$ (44)	
horizontal rectangle source in the xy plane at $z = z_0$, $x \in [0, x_0]$, $y \in [-y_0, y_0]$ (I_a [$ML^{-2}T^{-1}$] is the contaminant mass released per source area per unit time, in the semi- infinitive thickness aquifer.)	$C(x,y,z,t) = \frac{1}{8n\sqrt{\pi D_z}}\int_0^t I_a(t-\tau)\exp(-\lambda\tau)\left\{\exp\left[-\frac{(z-z_0)^2}{4D_z\tau}\right] + \exp\left[-\frac{(z+z_0)^2}{4D_z\tau}\right]\right\}$ $\times \left[\text{erfc}\frac{x-u\tau-x_0}{2\sqrt{D_x\tau}} - \text{erfc}\frac{x-u\tau}{2\sqrt{D_x\tau}}\right]\left[\text{erfc}\frac{y-y_0}{2\sqrt{D_y\tau}} - \text{erfc}\frac{y+y_0}{2\sqrt{D_y\tau}}\right]\frac{d\tau}{\sqrt{\tau}}$ $-\infty < x,y < \infty, 0 \leqslant z \leqslant b, t > 0$ (45)	

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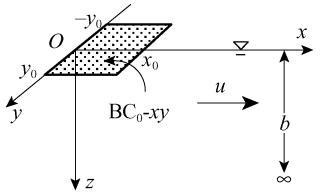
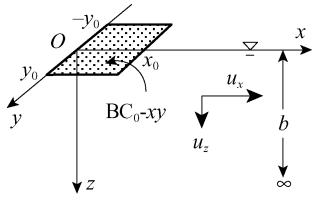
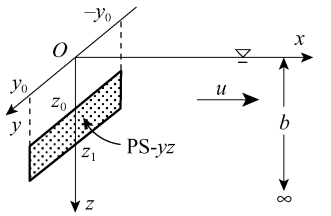
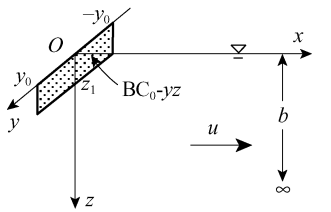
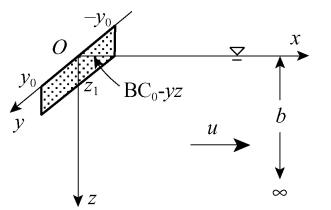
source type/boundary condition	solution	schematic diagram
horizontal boundary in the xy plane at $z = 0$ (The rectangle source is in $x \in [0, x_0]$, $y \in [-y_0, y_0]$. The aquifer is semi- infinitive in thickness; $C\|_{z=0} = \begin{cases} C_0, \text{when}(x,y) \in \text{source} \\ 0, \text{otherwise} \end{cases}$ where C_0 [ML^{-3}] is the source concentration.)[c]	$C(x,y,z,t) = \frac{C_0 z}{8\sqrt{\pi D_z}} \int_0^t \exp\left[-\lambda\tau - \frac{z^2}{4D_z\tau}\right] \times \left[\text{erfc}\frac{x-u\tau-x_0}{2\sqrt{D_x\tau}} - \text{erfc}\frac{x-u\tau}{2\sqrt{D_x\tau}}\right]\left[\text{erfc}\frac{y-y_0}{2\sqrt{D_y\tau}} - \text{erfc}\frac{y+y_0}{2\sqrt{D_y\tau}}\right]\frac{d\tau}{\tau^{3/2}}$ $-\infty < x,y < \infty, 0 < z \leqslant b, t > 0$ (46)	
horizontal boundary in the xy plane at $z = 0$ (The rectangle source is in $x \in [0, x_0]$, $y \in [-y_0, y_0]$. The aquifer is semi- infinitive in thickness; $C\|_{z=0} = \begin{cases} C_0, \text{when}(x,y) \in \text{source} \\ 0, \text{otherwise} \end{cases}$ where C_0 [ML^{-3}] is the source concentration; u_x, u_z are the horizontal and vertical velocities respectively.)	$C(x,y,z,t) = \frac{C_0 u_z}{4n} \int_0^t \exp(-\lambda\tau)\left[\text{erfc}\frac{x-u_x\tau-x_0}{2\sqrt{D_x\tau}} - \text{erfc}\frac{x-u_x\tau}{2\sqrt{D_x\tau}}\right] \times \left[\text{erfc}\frac{y-y_0}{2\sqrt{D_y\tau}} - \text{erfc}\frac{y+y_0}{2\sqrt{D_y\tau}}\right] \times \left[\frac{1}{\sqrt{\pi D_z\tau}}\exp\left(-\frac{(z-u_z\tau)^2}{4D_z\tau}\right) - \frac{u_z}{2D_z}\exp\left(\frac{u_z z}{D_z}\right) erfc\frac{z+u_z\tau}{2\sqrt{D_z\tau}}\right]d\tau$ $-\infty < x,y < \infty, 0 < z \leqslant b, t > 0$ (47)	
vertical rectangle source in the yz plane at $x = 0$, $y \in [-y_0, y_0]$, $z \in [z_0, z_1]$ (I_a [$ML^{-2}T^{-1}$] is the contaminant mass released per unit source area per unit time, in the semi-infinitive thickness aquifer.)	$C(x,y,z,t) = \frac{1}{8n\sqrt{\pi D_x}} \int_0^t I_a(t-\tau)\exp\left[-\lambda\tau - \frac{(x-u\tau)^2}{4D_x\tau}\right]\left[\text{erfc}\frac{y-y_0}{2\sqrt{D_y\tau}} - \text{erfc}\frac{y+y_0}{2\sqrt{D_y\tau}}\right] \times \left[\text{erfc}\frac{z-z_1}{2\sqrt{D_z\tau}} - \text{erfc}\frac{z-z_0}{2\sqrt{D_z\tau}} + \text{erfc}\frac{z+z_0}{2\sqrt{D_z\tau}} - \text{erfc}\frac{z+z_1}{2\sqrt{D_z\tau}}\right]\frac{d\tau}{\sqrt{\tau}}$ $-\infty < x,y < \infty, 0 \leqslant z \leqslant b, t > 0$ (48)	
vertical rectangle source in the yz plane at $x = 0$, $y \in [-y_0, y_0]$, $z \in [0, z_0]$ (The aquifer is semi- infinitive in thickness; $C\|_{x=0} = \begin{cases} C_0, \text{when}(y,z) \in \text{source} \\ 0, \text{otherwise} \end{cases}$ where C_0 [ML^{-3}] is the source concentration.)[d]	$C(x,y,z,t) = \frac{C_0 x}{8\sqrt{\pi D_x}} \int_0^t \exp\left[-\lambda\tau - \frac{(x-u\tau)^2}{4D_x\tau}\right] \times \left[\text{erfc}\frac{y-y_0}{2\sqrt{D_y\tau}} - \text{erfc}\frac{y+y_0}{2\sqrt{D_y\tau}}\right]\left[\text{erfc}\frac{z-z_1}{2\sqrt{D_z\tau}} - \text{erfc}\frac{z+z_1}{2\sqrt{D_z\tau}}\right]\frac{d\tau}{\tau^{3/2}}$ $x > 0, -\infty < y < \infty, 0 \leqslant z \leqslant b, t > 0$ (49)	
vertical boundary in the yz plane at $x = 0$ (The rectangle source is in $y \in [-y_0, y_0]$, $z \in [0, z_0]$; the aquifer is semi- infinitive in thickness; C_0 [ML^{-3}] is the source concentration. Note this is an approximate solution.)[e]	$C(x,y,z,t) = \frac{C_0}{8}\exp\left[\frac{ux}{2D_x}\left(1-\sqrt{1+\frac{4\lambda D_x}{u^2}}\right)\right]\text{erfc}\left[\frac{x-ut\sqrt{1+\frac{4\lambda D_x}{u^2}}}{2\sqrt{D_x t}}\right] \times \left[\text{erfc}\frac{y-y_0}{2\sqrt{D_y x/u}} - \text{erfc}\frac{y+y_0}{2\sqrt{D_y x/u}}\right]\left[\text{erfc}\frac{z-z_1}{2\sqrt{D_z x/u}} - \text{erfc}\frac{z+z_1}{2\sqrt{D_z x/u}}\right]$ $-\infty < x,y < \infty, 0 \leqslant z \leqslant b, t > 0$ (50)	

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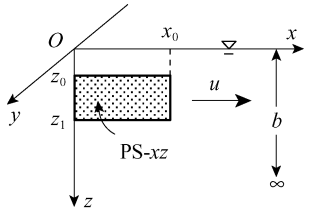
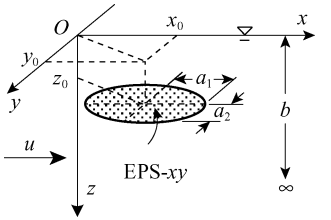
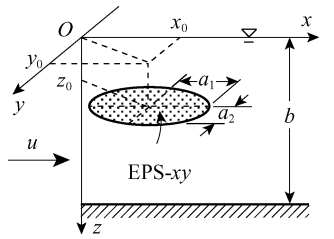
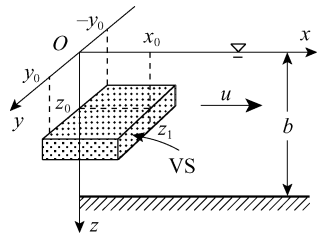
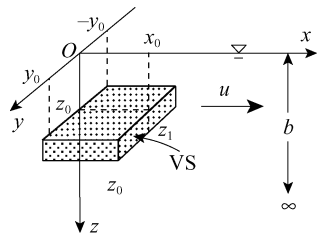
source type/boundary condition	solution	schematic diagram
	Steady solution along x- axis: $$C(x,0,0,\infty) = C_0 \exp\left[\frac{ux}{2D_x}\left(1-\sqrt{1+\frac{4\lambda D_x}{u^2}}\right)\right]\mathrm{erf}\left(\frac{y_0}{2\sqrt{D_y x/u}}\right)\mathrm{erf}\left(\frac{z_1}{2\sqrt{D_z x/u}}\right) \quad (51)$$	
vertical rectangle source in xz plane at $y = 0$, $x \in [0, x_0]$, $z \in [z_0, z_1]$ (I_a [$\mathrm{ML^{-2}T^{-1}}$] is the contaminant mass released per unit source area per unit time, in the semi-infinitive thickness aquifer.)	$$C(x,y,z,t) = \frac{1}{8n\sqrt{\pi D_y}}\int_0^t I_a(t-\tau)\exp\left(-\lambda\tau-\frac{y^2}{4D_y\tau}\right)\left[\mathrm{erfc}\frac{x-u\tau-x_0}{2\sqrt{D_x\tau}}-\mathrm{erfc}\frac{x-u\tau}{2\sqrt{D_x\tau}}\right]$$ $$\times\left[\mathrm{erfc}\frac{z-z_1}{2\sqrt{D_z\tau}}-\mathrm{erfc}\frac{z-z_0}{2\sqrt{D_z\tau}}+\mathrm{erfc}\frac{z+z_0}{2\sqrt{D_z\tau}}-\mathrm{erfc}\frac{z+z_1}{2\sqrt{D_z\tau}}\right]\frac{\mathrm{d}\tau}{\sqrt{\tau}}$$ $$-\infty < x,y < \infty, 0 \leqslant z < \infty, t > 0 \quad (52)$$	
horizontal elliptic source in the xy plane at $z = z_0$ with the center at (x_0, y_0, z_0), and a_1 and a_2 as the semi-axis paralleled respectively to x- and y-axis (I_a [$\mathrm{ML^{-2}T^{-1}}$] is contaminant mass released per source area per unit time. The aquifer is semi-infinitive in thickness. In the equation, I_1 is modified Bessel function of first kind of first order; λ, λ_s, [$\mathrm{T^{-1}}$] first-order constants for decay in liquid and sorbed phases; and r_1, r_2 [$\mathrm{T^{-1}}$] the forward and reverse rate coefficients respectively.)[f]	$$C(x,y,z,t) = \frac{1}{8\pi n\sqrt{D_x D_z}}\int_0^t\int_{x_0-a_1}^{x_0+a_1} I_a(t-\tau)F_1(\tau)\left\{\int_0^\tau F_2(\tau)F_3(x-q,p)F_4(p)F_5(p)\frac{\mathrm{d}p}{p}\right.$$ $$\left.+\frac{F_3(x-q,\tau)}{\tau}F_4(\tau)F_5(\tau)\right\}\mathrm{d}q\mathrm{d}\tau$$ $$-\infty < x,y < \infty, 0 \leqslant z < \infty, t > 0 \quad (53)$$ where $F_1(t) = \exp[-(r_2+\lambda_s)t]$; $F_2(t) = I_1\left(2\sqrt{r_1 r_2 p(t-p)}\right)\sqrt{\frac{r_1 r_2 p}{t-p}}$ $$F_3(x,t) = \exp\left[\frac{ux}{2D_x}-\frac{x^2}{4D_x t}-t\left(r_1+\lambda-r_2-\lambda_s+\frac{u^2}{4D_x}\right)\right]$$ $$F_4(t) = \exp\left[\frac{-(z+z_0)^2}{4D_z t}\right]+\exp\left[\frac{-(z-z_0)^2}{4D_z t}\right];$$ $$F_5(t) = \mathrm{erfc}[k_2(q,y,t)] - \mathrm{erfc}[k_1(q,y,t)]$$ $$k_1(q,y,t) = \frac{y-y_0+Y}{2\sqrt{D_y t}}$$ $$k_2(q,y,t) = \frac{y-y_0-Y}{2\sqrt{D_y t}}$$ $$y = a_2\sqrt{1-\frac{(q-x_0)^2}{a_1^2}}$$	

(*Continued*)

source type/boundary condition	solution	schematic diagram
horizontal elliptic source, as described above with the finite aquifer thickness, b[f]	$C(x,y,z,t)=\frac{1}{4nb\sqrt{\pi D_x}}\int_0^t\int_{x_0-a_1}^{x_0+a_1}I_a(t-\tau)F_1(\tau)\left\{\int_0^\tau F_2(\tau)F_3(x-q,p)F_5(p)F_6(p)\frac{\mathrm{d}p}{\sqrt{p}}+\frac{F_3(x-q,\tau)}{\sqrt{\tau}}F_5(\tau)F_6(\tau)\right\}\mathrm{d}q\mathrm{d}\tau$ $-\infty<x,y<\infty, 0\leqslant z\leqslant b, t>0$ (54) where $F_6(t)=1+2\sum_{n=1}^{\infty}\cos\frac{n\pi z_0}{b}\cos\frac{n\pi z}{b}\exp\left(-\frac{D_z n^2\pi^2}{b^2}t\right)$; the rest are as above	
parallelepiped body source at $x\in[0, x_0]$, $y\in[-y_0, y_0]$, $z\in[0, z_0]$ (I [$\mathrm{ML^{-3}T^{-1}}$] is the contaminant mass released per unit source volume per unit time.)[a]	$C(x,y,z,t)=\frac{1}{4nb}\int_0^t I(t-\tau)\exp(-\lambda\tau)\left[\mathrm{erfc}\frac{x-u\tau-x_0}{2\sqrt{D_x\tau}}-\mathrm{erfc}\frac{x-u\tau}{2\sqrt{D_x\tau}}\right]$ $\times\left[\mathrm{erfc}\frac{y-y_0}{2\sqrt{D_y\tau}}-\mathrm{erfc}\frac{y+y_0}{2\sqrt{D_y\tau}}\right]$ $\times\left[z_1-z_0+\frac{2b}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left(\sin\frac{m\pi z_1}{b}-\sin\frac{m\pi z_0}{b}\right)\cos\frac{m\pi z}{b}\exp\left(-\frac{D_z m^2\pi^2}{b^2}\tau\right)\right]\mathrm{d}\tau$ $-\infty<x,y<\infty, 0\leqslant z\leqslant b, t>0$ (55)	
parallelepiped body source, as described above, but with semi- infinitive aquifer thickness	$C(x,y,z,t)=\frac{1}{8n}\int_0^t I(t-\tau)\exp(-\lambda\tau)\left[\mathrm{erfc}\frac{x-u\tau-x_0}{2\sqrt{D_x\tau}}-\mathrm{erfc}\frac{x-u\tau}{2\sqrt{D_x\tau}}\right]$ $\times\left[\mathrm{erfc}\frac{y-y_0}{2\sqrt{D_y\tau}}-\mathrm{erfc}\frac{y+y_0}{2\sqrt{D_y\tau}}\right]$ $\times\left[\mathrm{erfc}\frac{z-z_1}{2\sqrt{D_z\tau}}-\mathrm{erfc}\frac{z-z_0}{2\sqrt{D_z\tau}}+\mathrm{erfc}\frac{z+z_0}{2\sqrt{D_z\tau}}-\mathrm{erfc}\frac{z+z_1}{2\sqrt{D_z\tau}}\right]\mathrm{d}\tau$ $-\infty<x,y<\infty, 0\leqslant z<\infty, t>0$ (56)	

Notes: [a] Park and Zhan, 2001 [20]; [b] Neville, 1994 [16]; [c] Leij et al., 2000 [19]; [d] Sagar, 1982[21]; [e] Domenico, 1987 [14]; [f] Sim et al., 1999 [25]. PtS: point source; LS-x, LS-y, and LS-z: line sources along directions x, y, and z respectively; PS$-xy$, PS$-xz$, and PS$-yz$: plane sources in planes xy, xz, and yz respectively; BC$_0-xy$ and BC$_0-yz$: given concentration boundaries in xy and yz planes respectively; EPS-xy: elliptic plane source in the xy plane; VS: volumetric source.

$$C(x,y,z,t) = \frac{M_p}{4\pi nbt\sqrt{D_xD_y}}\exp\left[-\lambda t - \frac{(x-ut)^2}{4D_x\tau} - \frac{y^2}{4D_yt}\right] \times\left[1+2\sum_{m=1}^{\infty}\cos\frac{m\pi z_0}{b}\cos\frac{m\pi z}{b}\exp\left(-\frac{D_zm^2\pi^2}{b^2}t\right)\right]. \quad (57)$$

For the cases of constant concentration boundaries, the instantaneous solutions can be obtained by taking out of the temporal integration, substituting t for τ, and multiplying the right-handed terms by t in the corresponding equations of given C_0. For instance, the instantaneous solution in accordance with Eq. (41) is:

$$C(x,y,z,t) = \frac{C_0x}{4b\sqrt{\pi D_xt}}\exp\left[-\lambda t - \frac{(x-ut)^2}{4D_xt}\right] \times\left[\operatorname{erfc}\frac{y-y_0}{2\sqrt{D_yt}} - \operatorname{erfc}\frac{y+y_0}{2\sqrt{D_yt}}\right] \times\left[z_1 - z_0 + \frac{2b}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left(\sin\frac{m\pi z_1}{b} - \sin\frac{m\pi z_0}{b}\right) \cos\frac{m\pi z}{b}\exp\left(-\frac{D_zm^2\pi^2}{b^2}t\right)\right]. \quad (58)$$

3.3 Solutions for short period release

The source strength for short period release can be expressed as

$$I(t) = \begin{cases} I_0f(t), 0 < t \leqslant t_0, \\ 0, t > t_0, \end{cases} \quad (59)$$

where t_0 [T] is the time of source elimination. When the calculation time $t \leqslant t_0$, the solutions are the same as those for the continuous release. When $t > t_0$, the solutions can be obtained by changing the lower limit of the integration from 0 to (t–t_0).

3.4 Solutions for the exponential decay of the source strength

In this case, the change of source strength with time is determinative, which can be expressed as $f(t) = \exp(-\alpha t)$, where α [T^{-1}] is the decay constant. Considering the problems of the point source, for instance, $I_p(t) = I_{p0}\exp(-\alpha t)$, where I_{p0} [MT^{-1}] is the initial point source strength. Substituting this function into Eq. (32) yields

$$C(x,y,z,t) = \frac{I_{p0}}{4\pi nb\sqrt{D_xD_y}}\int_0^t\exp[-\alpha(t-\tau)] \times\exp\left[-\lambda\tau - \frac{(x-u\tau)^2}{4D_x\tau} - \frac{y^2}{4D_y\tau}\right] \times\left[1+2\sum_{m=1}^{\infty}\cos\frac{m\pi z_0}{b}\cos\frac{m\pi z}{b}\exp\left(-\frac{D_zm^2\pi^2}{b^2}\tau\right)\right]\frac{d\tau}{\tau}. \quad (60)$$

3.5 Transport with retardation-sorption

If there exists solid sorption during the transport process as stated in the assumptions, and the sorption process can be represented by the linear module of Henry type, the governing equation is

$$R_d\frac{\partial C}{\partial t} = D_x\frac{\partial^2C}{\partial x^2} + D_y\frac{\partial^2C}{\partial y^2} + D_z\frac{\partial^2C}{\partial z^2} - u\frac{\partial C}{\partial x} - \lambda R_dC + \frac{I}{n}, \quad (61)$$

where λ is the rate constant of first-order decay in the dissolved and sorbed phases; and R_d is retardation factor

$$R_\mathrm{d} = 1 + \frac{\rho_\mathrm{b}}{n}K_\mathrm{d}, \quad (62)$$

where ρ_b [ML^{-3}] is porous medium bulk density, and K_d [L^3M^{-1}] is the distribution coefficient.

If

$$D_x' = D_x/R_\mathrm{d},\ D_y' = D_y/R_\mathrm{d},\ D_z' = D_z/R_\mathrm{d}, \quad u' = u/R_\mathrm{d},\ I' = I/R_\mathrm{d},\ M' = M/R_\mathrm{d}, \quad (63)$$

in Eq. (61), the equation is identical to Eq. (1) in form, which means that its solutions are the same if the above replacements are substituted. Hence, all solutions given above are applicable for the retardation transport, only if D_x, D_y, D_z, u, I, and M in the equations are replaced by D_x', D_y', D_z', u', I', and M', respectively.

If the decay in the sorbed phase is not neglectable, the governing equation becomes

$$R_\mathrm{d}\frac{\partial C}{\partial t} = D_x\frac{\partial^2C}{\partial x^2} + D_y\frac{\partial^2C}{\partial y^2} + D_z\frac{\partial^2C}{\partial z^2} - u\frac{\partial C}{\partial x} - \lambda C + \frac{I}{n}. \quad (64)$$

In this case, λ must also be replaced by $\lambda' = \lambda/R_\mathrm{d}$.

3.6 Solutions for the sources near linear boundaries

Two principal types of linear boundaries are often encountered in felds for the contaminant transport studies: impenetrable boundary, such as impermeable layers; and absorbable boundary, such as rivers connected to aquifers. It is supposed that the contaminant can not penetrate through the impermeable layers. The river can eliminate the contaminant instantaneously when it flows into the river. For these cases, if ground water flows along the boundary direction, the solutions for the concentration can be obtained using the method of imaging when the contaminant has reached the boundaries, as shown in Fig. 1. The coordinates of calculation point p are (x, y) in xOy system (real domain) and (ξ, ζ) in $\xi o\zeta$ system

(imaging domain), respectively. Substituting them into Eq. (55) yields two solutions C_1 and C_2. For the impenetrable boundary, the concentration for point p is $C = C_1 + C_2$; for the river boundary, $C = C_1 - C_2$. The method of imaging are often used for the solutions of horizontally infinitive domain, that is, $-\infty < x,y < +\infty$.

3.7 Superposition of multiple sources and source changes with time

The solutions discussed above can also be used for concentration calculation for multiple sources. On the basis of superposition principles, the concentration for calculation point p is equal to the sum of concentrations for individual sources. This can also be used for the irregular source problems, since an irregular source can be approximated by the combination of a limited number of regular sources. Similarly, if the changes of source strength or concentration with time can be represented by a limited number of steps, the superimposed solutions can also be obtained [1].

3.8 Dimensionless solutions

Dimensionless parameters are particularly useful for drawing type curves and comparing analytical solutions, because a much wider range of parameter values can be covered in a compact fashion than the situation when dimensional parameters are used [29].

The dimensionless parameters can be defined differently for different kinds of solutions [4,20,29]. In this paper, the dimensionless parameters are defined as follows:

$$C_{\mathrm{D}} = \begin{cases} C(x, y, z, t)/C_0, \text{ for solutions of given boundary concentration } C_0, \\ C(x, y, z, t)nb\sqrt{D_xD_y}/I_0, \text{ for solutions of given source strength } I_0, (b = \text{constant}), \\ C(x, y, z, t)nD_x\sqrt{D_yD_z}/I_0u, \text{ for solutions of given source strength } I_0, (b \to +\infty) \end{cases} \tag{65}$$

$$t_{\mathrm{d}} = \frac{ut}{x}, x_{\mathrm{d}} = \frac{ux}{D_x} = Pe, x_{\mathrm{d0}} = \frac{ux_0}{D_x} = Pe_0, \quad y_{\mathrm{d}} = \frac{uy}{\sqrt{D_xD_y}}, y_{\mathrm{d0}} = \frac{uy_0}{\sqrt{D_xD_y}}, \lambda_{\mathrm{d}} = \frac{\lambda D_x}{u^2}. \tag{66}$$

For the finite thickness (b = constant),

$$z_{\mathrm{d}} = \frac{z}{b}, z_{\mathrm{d0}} = \frac{z_0}{b}, z_{\mathrm{d1}} = \frac{z_1}{b}, u_{\mathrm{d}} = \frac{bu}{\sqrt{D_xD_z}}. \tag{67}$$

For the semi-infinitive or infinitive thickness ($b \to +\infty$ or $b \to \pm\infty$)

$$z_{\mathrm{d}} = \frac{uz}{\sqrt{D_xD_z}}, z_{\mathrm{d0}} = \frac{uz_0}{\sqrt{D_xD_z}}, z_{\mathrm{d1}} = \frac{uz_1}{\sqrt{D_xD_z}}. \tag{68}$$

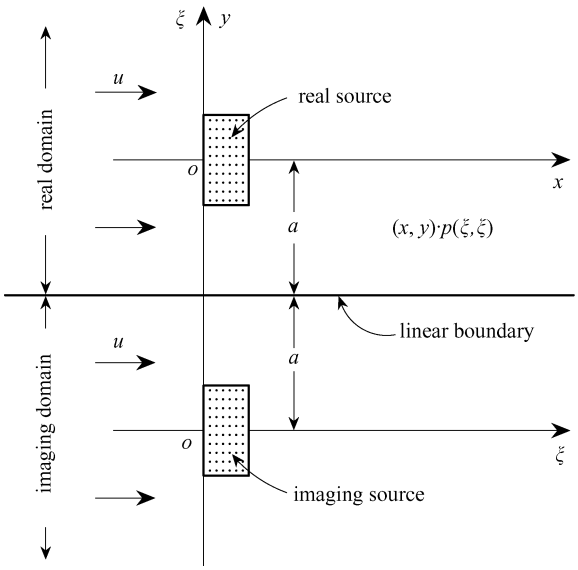


Fig. 1 Schematic representation of imaging method near a linear boundary

In the above equations, C_{D} is relative concentration, t_{D} is dimensionless time, x_{D}, y_{D}, and z_{D} are dimensionless distances in directions x, y, and z, respectively, Pe is the Peclet number, λ_{D} is the dimensionless decay constant, u_{D} is the dimensionless pore velocity, and I_0 [$\mathrm{MT^{-1}}$] is the source mass released per unit time. It is not feasible to compare the dimensionless calculation results for different definitions, since the dimensionless parameters are different either for an infinitive and finite aquifer or for a given source strength or concentration.

With dimensionless parameters as defined above, all solutions in the library can be transferred into their dimensionless forms. For instance, Eqs. (32) and (33) in Table 3 become

$$C_{\mathrm{d}} = \frac{1}{4\pi}\int_0^{t_D} \exp\left[-\lambda_D Pe\tau - \frac{Pe(1-\tau)^2}{4\tau} - \frac{y_{\mathrm{d}}^2}{4pe\tau}\right] \times \left[1 + 2\sum_{m=1}^{\infty}\cos m\pi z_{\mathrm{d0}}\cos m\pi z_{\mathrm{d}}\exp\left(-\frac{m^2\pi^2}{u_{\mathrm{d}}^2}Pe\tau\right)\right]\frac{\mathrm{d}\tau}{\tau}, \tag{69}$$

and

$$C_{\mathrm{d}} = \frac{1}{8\sqrt{\pi^3 Pe}}\int_0^{t_D} \exp\left[-\lambda_{\mathrm{d}} Pe\tau - \frac{Pe(1-\tau)^2}{4\tau} - \frac{y_{\mathrm{d}}^2}{4pe\tau}\right] \times \left\{\exp\left[-\frac{(z_{\mathrm{d}} - z_{\mathrm{d0}})^2}{4Pe\tau}\right] + \exp\left[-\frac{(z_{\mathrm{d}} + z_{\mathrm{d0}})^2}{4Pe\tau}\right]\right\}\frac{\mathrm{d}\tau}{\tau^{3/2}}, \tag{70}$$

where τ is the dimensionless integration variable (t_{D}). Equation (49), which is a solution for given boundary concentration C_0, becomes

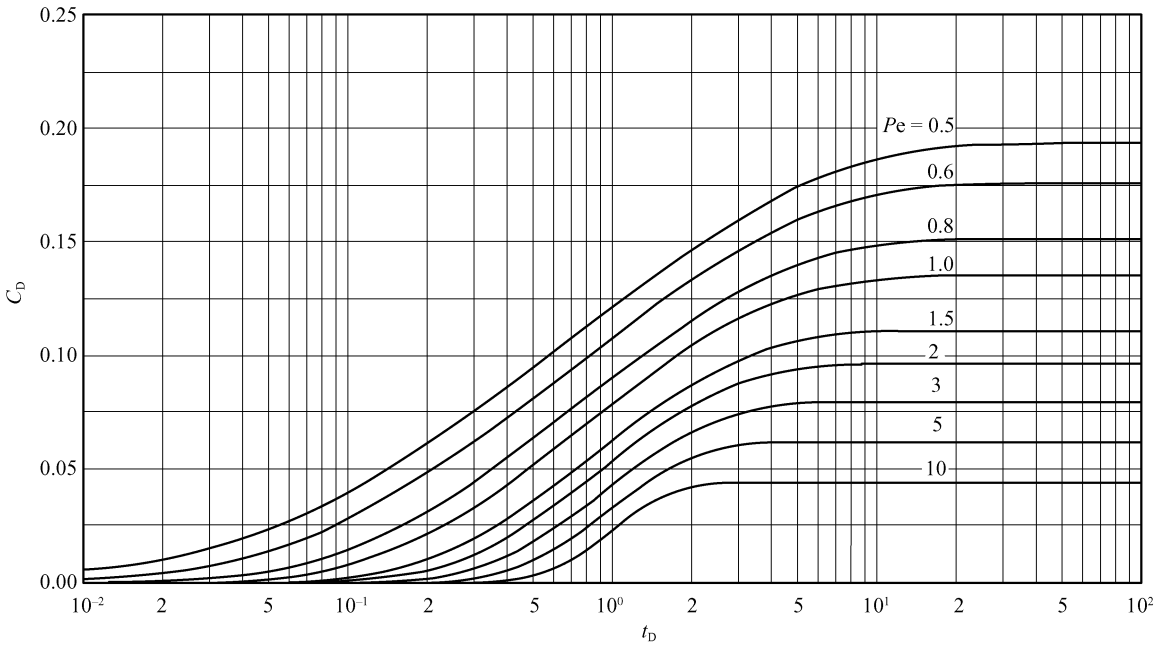


Fig. 2 Type curves of Eq. (72) along the plume centerline for different values of Peclet number (*Pe*)

$$C_D = \frac{1}{8}\sqrt{\frac{Pe}{\pi}}\int_0^{t_D}\exp\left[-\lambda_D Pe\tau - \frac{Pe(1-\tau)^2}{4\tau}\right] \times\left[\operatorname{erfc}\frac{y_d - y_{d0}}{2\sqrt{Pe\tau}} - \operatorname{erfc}\frac{y_D + y_{D0}}{2\sqrt{Pe\tau}}\right]\left[\operatorname{erfc}\frac{z_D - z_{D1}}{2\sqrt{Pe\tau}} - \operatorname{erfc}\frac{z_D - z_{D0}}{2\sqrt{Pe\tau}}\right]\frac{d\tau}{\tau^{3/2}}. \quad (71)$$

4 Example calculations

The analytical solution library presented above can be widely applied for various practical purposes. Here we will discuss two kinds of applications: one is drawing type curves, which can be used, for instance, for parameter identification; and the other is comparing the effect of source geometries on concentration calculations.

4.1 Examples of type curves

Eq. (55) is taken as an example to draw the type curves. Its dimensionless form is

$$C_D = \frac{Pe}{8y_{d0}(z_{D1} - z_{D0})Pe_0}\int_0^{t_D}\exp(-\lambda_D Pe\tau) \times\left[\operatorname{erfc}\frac{Pe(1-\tau) - Pe_0}{2\sqrt{pe\tau}} - \operatorname{erfc}\frac{Pe(1-\tau)}{2\sqrt{Pe\tau}}\right]\left[\operatorname{erfc}\frac{y_D - y_{D0}}{2\sqrt{Pe\tau}} - \operatorname{erfc}\frac{y_D + y_{D0}}{2\sqrt{Pe\tau}}\right]\left[\ z_{D1} - z_{D0} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}(\sin m\pi z_{d1} - \sin m\pi z_{D0})\cos m\pi z_D \exp\left(-\frac{m^2\pi^2}{u_D^2}Pe\tau\right)\right]d\tau. \quad (72)$$

Figure 2 presents a graph of relative concentrations (C_D) versus dimensionless time (t_D), calculated using Eq. (72), without decay ($\lambda_D = 0$) or sorption ($R_D = 1$) and for different values of Peclet numbers (*Pe*). Calculations are performed along the plume centerline ($y_D = 0$, $z_D = 0.5$) for a source of coordinates $x_{D0} = 0.5$, $y_{D0} = 0.25$, $z_{D0} = 0$, $z_{D1} = 0.5$, and $u_D = 1$. It is found that dimensionless time values to reach quasi steady-state conditions are different for different Peclet numbers and smaller Peclet numbers have larger dimensionless time values to reach. This is different than what was described by Guyonnet and Neville (2004), in which a constant dimensionless time value of 10 is suggested [29]. From Fig. 2 it can be seen that the concentrations increase with time for agiven space point. Different places have experienced a similar concentration change. Locations of smaller Peclect numbers ($Pe = x_D$) have longer concentration increase duration of dimentionless time than those of larger ones before they reach their steady state.

Figure 3 is the calculation results of Eq. (72) for steady state concentrations along the x direction for different values of dimensionless lateral distance (y_D) with z_D kept equal to 0.5 and u_D to 1. The calculations are performed with the same source conditions as in Fig. 2 with y_D values from 0 to 5. As groundwater flows along the x axis and the dimensionless source length is 0.5 in the x direction ($Pe_0 = x_{D0}$), the steady concentrations increase in the places of Peclet number from 0 to 3 along the plume centerline with the maximum value at the point of Peclet number equal to 0.3. Then the concentrations decrease rapidly. The steady concentrations are significantly influenced by lateral distance to the plume centerline, especially at low values of Peclet number, but with a similar trend of distributions.

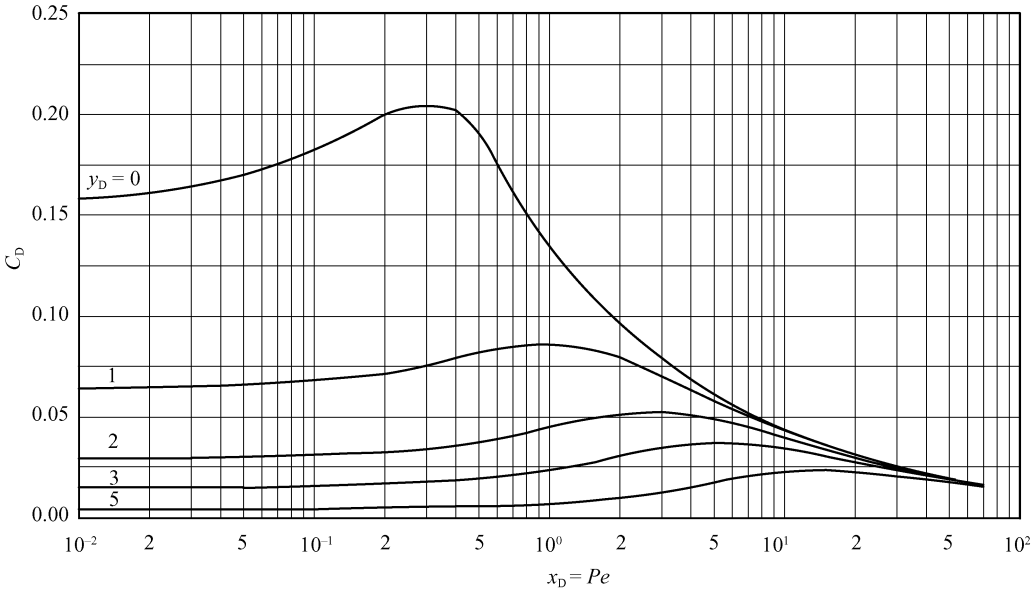


Fig. 3 Steady state concentrations of Eq. (72) along the x direction for different values of dimensionless lateral distance (y_D) with $z_D = 0.5$ and $u_D = 1$

4.2 Effect of source geometries on the concentrations

For a given transport equation, analytical solutions will differ according to the source geometry [29]. Huyakorn et al. [31] compared several analytical solutions for three-dimensional transport in groundwater. Guyonnet and Neville [29] compared two solutions: one is the solution proposed by Domenico [14], as shown in Eq. (50) and the other, by Sagar [21] in Eq. (49). As an example, the effect of three different sources—point, line, and plane on the concentration calculations will be discussed. The dimensionless forms of Eq. (36) for the line source and Eq. (40) for the plane source are

$$C_D = \frac{1}{8y_{D0}}\sqrt{\frac{Pe}{\pi}}\int_0^{t_D}\exp\left[-\lambda_D Pe\tau - \frac{Pe(1-\tau)^2}{4\tau}\right] \times\left[\operatorname{erfc}\frac{y_D - y_{D0}}{2\sqrt{Pe\tau}} - \operatorname{erfc}\frac{y_D + y_{D0}}{2\sqrt{Pe\tau}}\right] \times\left[1 + 2\sum_{m=1}^{\infty}\cos m\pi z_{D0}\cos m\pi z_D \exp\left(-\frac{m^2\pi^2}{u_D^2}Pe\tau\right)\right]\frac{d\tau}{\sqrt{\tau}}, \tag{73}$$

and

$$C_D = \frac{1}{8y_{D0}(z_{D1} - z_{D0})}\sqrt{\frac{Pe}{\pi}}\int_0^{t_D}\exp\left[-\lambda_D Pe\tau - \frac{Pe(1-\tau)^2}{4\tau}\right] \times\left[\operatorname{erfc}\frac{y_D - y_{D0}}{2\sqrt{Pe\tau}} - \operatorname{erfc}\frac{y_D + y_{D0}}{2\sqrt{Pe\tau}}\right] \times\left[z_{D1} - z_{D0} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}(\sin m\pi z_{D1} - \sin m\pi z_{D0}) \times\cos m\pi z_D \exp\left(-\frac{m^2\pi^2}{u_D^2}Pe\tau\right)\right]\frac{d\tau}{\sqrt{\tau}}, \tag{74}$$

respectively, and the dimensionless form for the point source is shown in Eqs (69) and (70). The sources are placed in the middle of the aquifer thickness with a dimensionless length of 1.0 for the line source and side lengths of 1.0×0.5 for the rectangle plane source. Three types of sources have the same source strength. The concentrations are calculated for the steady state along the plume centerline and the line of $y_D = 0.5$ and $z_D = 0.25$, as shown in Fig. 4.

It is shown that the source types affect the concentration distribution significantly for smaller values of Peclet number. However, this effect decreases with the increase in Peclet number. For the case described above with unit aquifer thickness and isotropic dispersion coefficients, the concentration difference is less than 5% when the Peclet number is greater than 1.0. As the source size gets larger, the medium gets more anisotropic. When the aquifer gets thicker, the effect of source types on concentrations becomes greater. Accordingly, the value of Peclet number, beyond which the effect of source types can be neglected, gets larger.

5 Conclusions

The library of analytical solutions presented in this paper is a useful tool to analyze the contaminant transport in porous media. Dimensionless forms of the solutions provided make dimensionless analyses very convenient, such as drawing type curves and comparing solutions. The correctness of the solutions is well tested by known solutions and inter-comparison. As the library covers most of the available solutions, it can be used to solve a wide range of contaminant transport problems for one or multiple

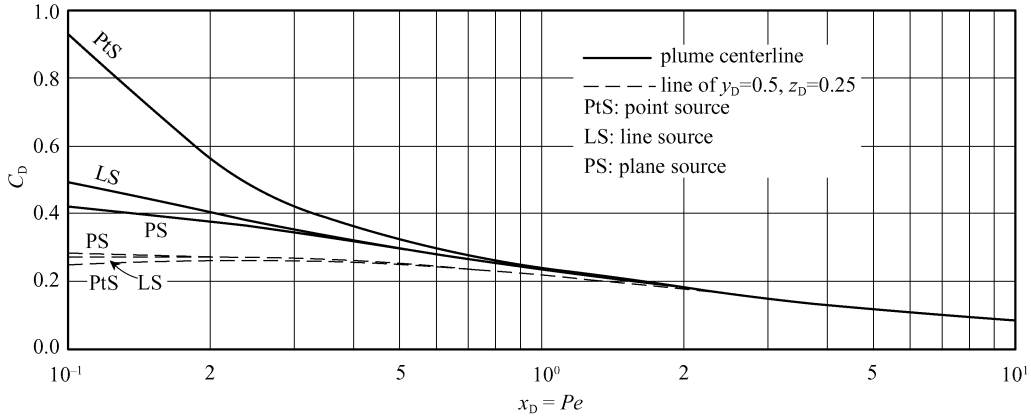


Fig. 4 Comparison of steady state concentrations along the x direction for the point, line and plane sources

dimensions. By using the Green's function method, solutions for specific problems can be obtained by multiplying the directional solutions and their integration results as listed in Tables 1 and 2. Users can select the solutions in the library or construct their own solutions to solve their specific problems.

It should be noted that the solutions in the library are obtained under a series of assumptions. The most important assumptions are homogeneous media, constant dispersion coefficients, steady uniform flow, first-order decay, and linear sorption. Problems of variable dispersion coefficients and nonuniform flow such as radius flow are excluded in the library.

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